Quantitative Social Choice Theory

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What is Social Choice Theory ?

- Social choice theory is the theory of collective decision making.
- Major results in economics in the 50-70s.
- Some of the most celebrated results in the theory are negative:
- Arrow (1961) proved "irrationality" of ranking 3 or more alternatives and
- Gibbard–Satterthwaite (1973) proved that electing among 3 or more alternatives can always be manipulated.
- These and other irrationality result contributed to popularity of mechanism designs.



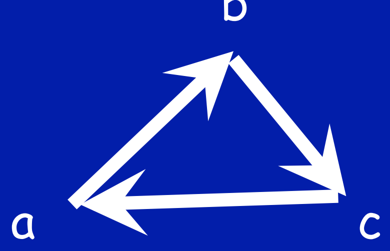
What is Quantitative Social choice?

- In quantitative social choice – take a second look at these questions.
- <u>Basic premise</u>: Is it possible to avoid non-rationality or manipulation with very good probability?

- The answer to this question is typically yes if there is a strong bias towards a certain alternatives.
- <u>We assume:</u> large number of voters / alternatives. And uniform distribution.
- Uniform distribution "stress-tests" the voting method.
- In this talk : a quantitative study of Arrow's theorem.

Condorcet Paradox

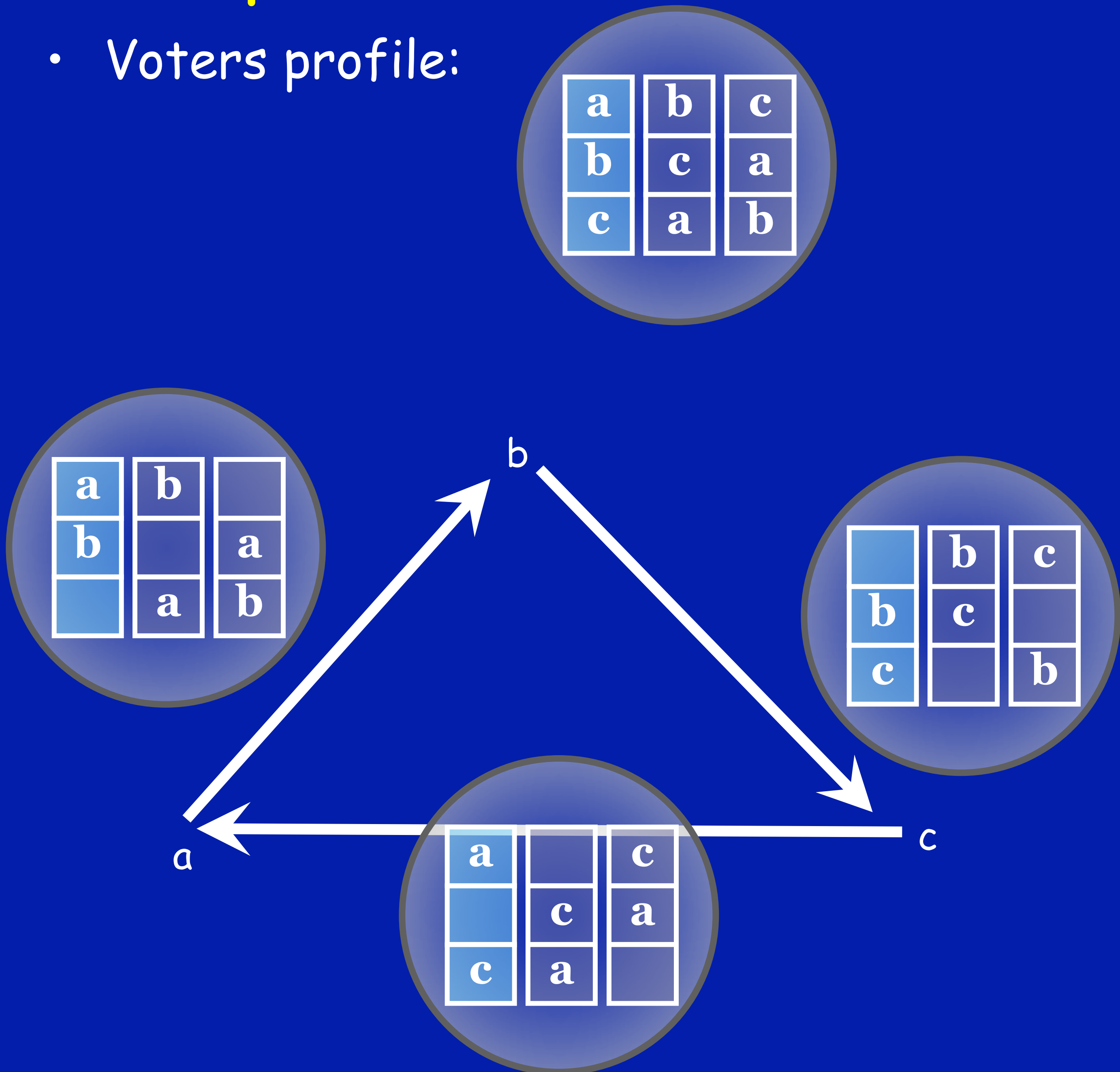
- n voters are to choose between 3 alternatives.
- Condorcet: Is there a rational way to do it?
- More specifically, for majority vote:
- Could it be that all of the following hold:
  - Majority of voters rank a above b?
  - Majority of voters rank b above c?
  - Majority of voters rank c above a?



- Defined by Marquis de Condorcet (1785) as part of a discussion of the best way to rank candidates to the French academy of Sciences.

Example of a Condorcet Paradox

- Voters profile:

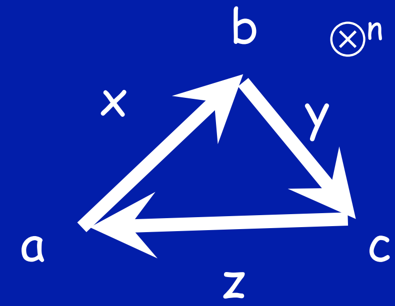


Majority and Other Functions …

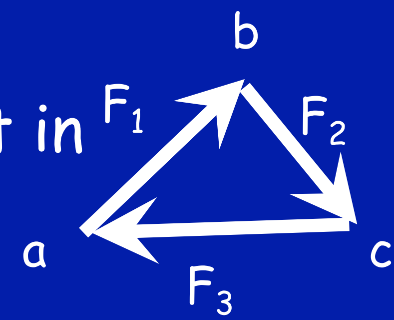
- Question: Why did we get in irrational outcome?
- Is this a problem with the majority function?
- What if the choice between any two candidates is decided by some other function?
- Say, an electoral college?
- To answer these questions, first some notation

Properties of Constitutions

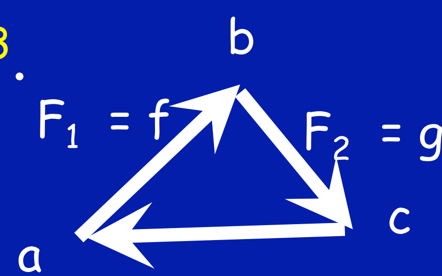
- n voters are to choose between 3 alternatives
- Voter i ranking := $\sigma_i \in S(3)$. Let:



- $x_i = +1$ if $\sigma_i(a) > \sigma_i(b)$, $x_i = -1$ if $\sigma_i(a) < \sigma_i(b)$,
- $y_i = +1$ if $\sigma_i(b) > \sigma_i(c)$, $y_i = -1$ if $\sigma_i(b) < \sigma_i(c)$,
- $z_i = +1$ if $\sigma_i(c) > \sigma_i(a)$, $z_i = -1$ if $\sigma_i(c) < \sigma_i(a)$.
- <u>Note:</u> (x_i,y_i,z_i) correspond to a σ_i iff (x_i,y_i,z_i) not in $\{(1,1,1),(-1,-1,-1)\}$



- <u>Def:</u> A <u>constitution</u> is a map $F : S(3)^n \rightarrow \{-1,1\}^3$.
- <u>Def:</u> A constitution is <u>transitive</u> if for all σ:
  - $F(\sigma) \in \{-1,1\}^3 \setminus \{(1,1,1),(-1,-1,-1)\}$

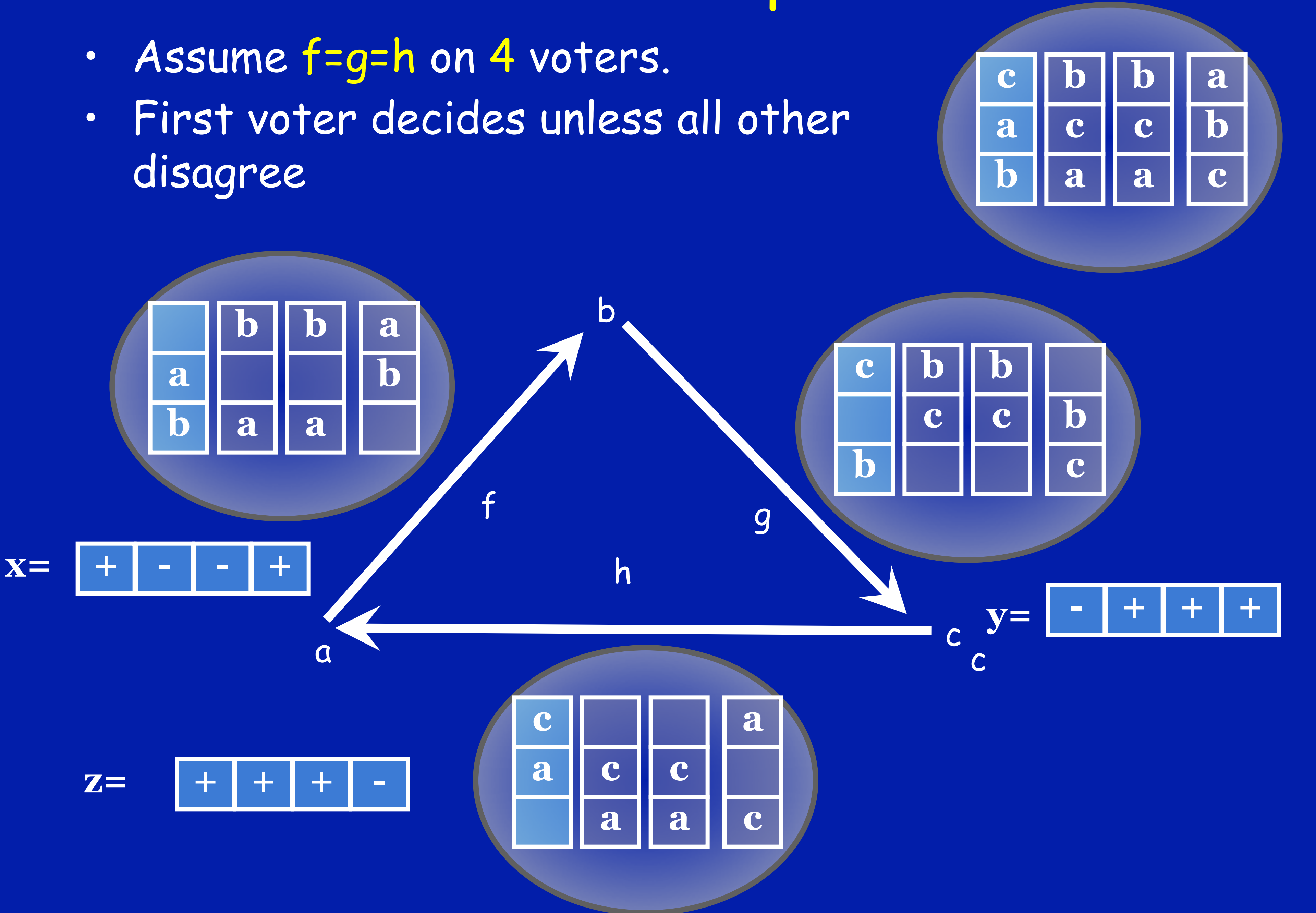


- <u>Def: Independence of Irrelevant Alternatives (IIA)</u> is satisfied by F if $\exists$ f,g,h s.t.:

  $F(\sigma) = (f(x(\sigma)),g(y(\sigma)),h(z(\sigma)))$ for all σ.

A second example

- Assume f=g=h on 4 voters.
- First voter decides unless all other disagree



Arrow's Impossibility Thm

Def: A constitution F satisfies Unanimity if

$\sigma_1 = \ldots = \sigma_n \Rightarrow F(\sigma_1,\ldots,\sigma_n) = \sigma_1$

(if all voters agree then this is the outcome)

- Thm (Arrow's "Impossibility", 61): Any constitution F on 3 (or more) alternatives which satisfies
- IIA,
- Transitivity and
- Unanimity:

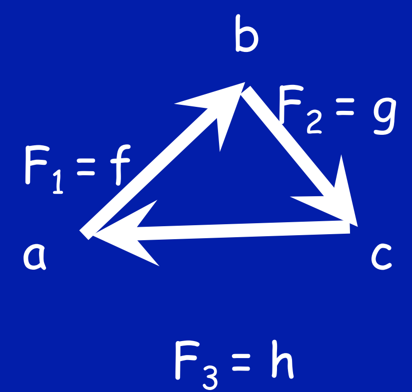
Is a dictator: There exists an i such that:

$F(\sigma) = F(\sigma_1,\ldots,\sigma_n) = \sigma_i$ for all σ

Arrow received the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel in 1972

A Short Proof of Arrow Thm

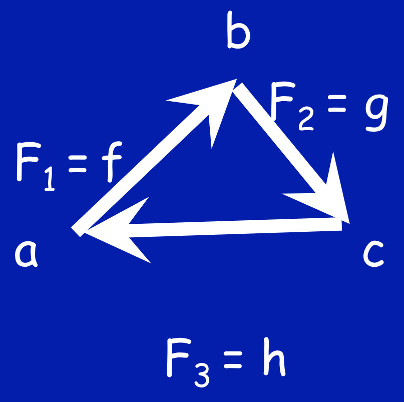
- Def: Voter 1 is pivotal for f (denoted $I_1(f) > 0$) if: $f(-,x_2,\dots,x_n) \neq f(+,x_2,\dots,x_n)$ for some $x_2,\dots,x_n$. (similarly for other voters).
- Barbera's Lemma (82, M-10): Any constitution $F=(f,g,h)$ on 3 alternatives which satisfies IIA and has
- $I_1(f) > 0$ and $I_2(g) > 0$
- has a non-transitive outcome.
- Pf: exist $x_2,\dots,x_n$ and $y_1,y_3,\dots,y_n$ s.t:
- $f(+1,+x_2,+x_3\dots,+x_n) \neq f(-1,+x_2,+x_3\dots,+x_n)$
- $g(+y_1,+1,+y_3,\dots,+y_n) \neq g(+y_1,-1,+y_3,\dots,+y_n)$
- $h(-y_1,-x_2,-x_3,\dots,-x_n) := v$ and choose x_1,y_2 s.t.: $f(x) = g(y) = v$ $\Rightarrow$ outcome is not transitive.
- Note: $(x_1,y_1,-y_1),(x_2,y_2,-x_2),(x_i,y_i,-x_i)$ not in $\{(1,1,1),(-1,-1,-1)\}$



A Short Proof of Arrow Thm

- Barbera's Pf of Arrow Thm (as carried in M'09)
- Let $F = (f,g,h)$.
- Let $I(f)$ = {pivotal voters for f}.
- Unanimity $\Rightarrow$ f,g,h are not constant

  $\Rightarrow$ $I(f), I(g), I(h)$ are non-empty.
- By Transitivity + lemma $\Rightarrow$ $I(f) = I(g) = I(h) = \{i\}$ for some i.
- $\Rightarrow F(\sigma) = G(\sigma_i)$
- By unanimity $\Rightarrow F(\sigma) = \sigma_i$.



A General Arrow Theorem

- Def: Write $A >_F B$ if for all σ and all $a \in A$ and $b \in B$ it holds that $F(\sigma)$ ranks a above b.
- Thm (M'10, see also Wilson-72): A constitution F on k alternatives satisfies IIA and Transitivity iff
- F satisfies that there exists a partition of the k alternatives into sets $A_1,\ldots A_s$ s.t:
- $A_1 >_F \ldots >_F A_s$ and
- If $|A_r| > 2$ then F restricted to A_r is a dictator on some voter j.

- Note: "Dictator" now is also $F(\sigma) = -\sigma$.
- Def: Let $F_k(n)$:= The set of constitutions on n voters and k alternatives satisfying IIA and Transitivity.

Random Rankings:

- Garman-Kamien 68 : Consider people voting according to a random order on a,b and c.

- Note: Rankings are chosen uniformly in S_3^n
- Assume IIA: $F(\sigma) = (f(x),g(y),h(z))$
- Q: What is the probability of a paradox:
- Def: $PDX(F) = P[f(x) = g(y) = h(z)]$?
- Arrow Theorem implies: If $F \neq$ dictator and f,g,h are non-constant then: $PDX(f) \geq 6^{-n}$.
- If Paradox unlikely perhaps do not care?
- Notation: Write $D(F,G) = P(F(\sigma) \neq G(\sigma))$.

<u>Probability of a Paradox</u>

- <u>Kalai-02:</u> If IIA holds with $F = (f,g,h)$ and
- $E[f] = E[g] = E[h] = 0$ then
- $PDX(F) < \varepsilon \Rightarrow \exists$ a dictator i s.t.:
- $D(F,\sigma_i) < K\,\varepsilon$ or $D(F,-\sigma_i) < K\,\varepsilon$
- Where K is some absolute constant.
- <u>Keller-08:</u> Same result for symmetric distributions.

- <u>However:</u>
- Proof does not work for other biases.
- Does not work for more than 3 alternative.
- Pf does not give a new proof of Arrow Theorem.

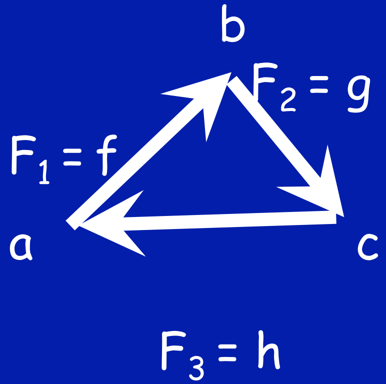
Probability of a Paradox

- General Thm M-10: For
- All number of alt. k, for all $\varepsilon > 0$, there exists $\delta > 0$ (δ does not depend on the number of voters n) s.t.:
- If IIA holds for F on k alternatives and n voters and
- $\min\{D(F,G) : G \in F_k(n)\} > \varepsilon$ (F is not close to $F_k(n)$)
- Then: $P(F) > \delta$ (Prob. of Paradox at least δ)

- Moral: Under the uniform distribution:
- Arrow's impossibility holds with good probability.
- Probability doesn't save us from irrationality no matter how large is the number of voters.
- Result may be viewed as a "testing" result.
- Comment: Can take $\delta = k^{-2} \exp(-C/\varepsilon^{21})$

A Quantitative Lemma from Proof

- Def: The influence of voter 1 on f (denoted $I_1(f)$) is:
- $I_1(f) := P[f(-,x_2,\ldots,x_n) \neq f(+,x_2,\ldots,x_n)]$
- Lemma (M-10): Any constitution $F=(f,g,h)$ on 3 alternatives which satisfies IIA and has
- $I_1(f) > \varepsilon$ and $I_2(g) > \varepsilon$
- Satisfies $PDX(F) > \varepsilon^3/36$.



- Pf:
- Let $A_f = \{x_3,\ldots,x_n : 1$ is pivotal for $f(*,*,x_3,\ldots,x_n)\}$
- Let $B_g = \{y_3,\ldots,y_n : 2$ is pivotal for $g(*,*,y_3,\ldots,y_n)\}$
- Then $P[A_f] > \varepsilon$ and $P[B_g] > \varepsilon$
- By "Inverse Hyper-Contraction": $P[A_f \cap B_g] > \varepsilon^3$.
- By Lemma: $PDX[F] \geq 1/36 \; P[A_f \cap B_g] > \varepsilon^3/36$.

Inverse Hyper Contraction

The Use of Swedish Technology

IKEA Store Falls Apart! Experts Blame Cheap Parts, Confusing Blueprint
From SD Headliner, Mar 25, 09.

Inverse Hyper Contraction

- <u>Note:</u> (x_i,y_i) are i.i.d. with $E(x_i,y_i) = (0,0)$ and $E[x_i\, y_i] = -1/3$
- <u>Results of C. Borell 82:</u> $\Rightarrow$
- Let $f,g : \{-1,1\}^n \to R_+$ then
- $E[f(x)\, g(y)] \geq |f|_p\, |g|_q$ if $1/9 \leq (1-q)\,(1-p)$ and $p,q < 1$.
- In particular: taking f and g indicators obtain:
- $E[f] > \varepsilon$ and $E[g] > \varepsilon \Rightarrow E[fg] > \varepsilon^3$.
- Implications in: M-O'Donnell-Regev-Steif-Sudakov-06.

- <u>Note:</u> "usual" hyper-contraction gives:
- $E[f(x)\, g(y)] \leq |f|_p\, |g|_q$ for all functions if
- $(p-1)(q-1) \geq 1/9$ and $p,q>1$.

Quantitative Arrow – 1st attempt

- Thm M-10: $\forall\ \varepsilon, \exists\ \delta$ s.t if IIA holds with $F = (f,g,h)$ &
- $\max\{|E[f]|, |E[g]|, |E[h]|\} < 1-\varepsilon$ &
- $\min\{D(F,G) : G \in F_3(n)\} > 3\varepsilon$
- Then $PDX(F) > (\varepsilon/96n)^3$.
- Pf Sketch: $P_f = \{i : I_i(f) > \varepsilon\, n^{-1}/4\}$ = f pivotal voters
- Since f is not almost constant ($\sum I_i(f) > Var[f] > \varepsilon/2$), P_f is not empty.
- If there exists $i \neq j$ with $i \in P_f$ and $j \in P_g$ then $PDX(F) > (\varepsilon/96n)^3$ by quantitative lemma.
- Otherwise $P_f = P_g = P_h = \{1\}$ and $P(F) < (\varepsilon/96n)^3$
- $\Rightarrow$ f,g and h are ε close to functions of x_1,y_1,z_1
- $\Rightarrow$ F is $3\ \varepsilon$ close to $G(\sigma) = G(\sigma_1)$, a function of voter 1.
- $PDX(G) \leq 3\ \varepsilon + (\varepsilon/96n)^3 < 1/6 \Rightarrow G \in F_3(n)$.

Quantitative Arrow – Real Proof

- Pf High Level Sketch:
- Let $P_f = \{i : I_i(f) > \varepsilon\}$.
- If there exists $i \neq j$ with $i \in P_f$ and $j \in P_g$ then $PDX(F) > \varepsilon^3 / 36$ by quantitative lemma.
- Two other cases to consider:
- I. $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty
- In this case: use Invariance + Gaussian Arrow Thm.
- II. $P_f \cup P_g \cup P_h = \{1\}$.
- In this case we condition on voter 1 so we are back in case I.

Quantitative Arrow – Real Proof

- The Low Influence Case:
- We want to prove the theorem under the condition that $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty.
- Let's first assume that $P_f = P_g = P_h$ is empty – all functions are influence at most I.
- Kalai noted that:
- $PDX(F) = \frac{1}{4}(1 - E[f(x)g(y)] - E[f(x)h(z)] - E[g(y)h(z)])$
- Where now (X,Y) is distributed as:
- $E[X_i] = E[Y_i] = 0$ and $E[X_i Y_i] = +1/3$
- By Majority is Stablest (M-Odonnell-Oleskewisz):
- $PFX(F) > PDX(u,v,w)$ + error(I) where
- $u(x) = sgn(\sum x_j + u_0)$ and $E[u] = E[f]$ etc.

Quantitative Arrow – Real Proof

- By Majority is Stablest:
- PFX(F) > PDX(u,v,w) + error(I) where
- $u(x) = \mathrm{sgn}(\sum x_j + u_0)$ and $E[u] = E[f]$ etc.
- Remains to bound PDX(u,v,w)
- By CLT this is approximately:
- $P[U>0, V>0, W>0] + P[U<0, V<0, W<0]$ where $U \sim N(E(u), 1)$, $V \sim N(E(v),1)$ and $W \sim N(E(w),1)$ &
- $Cov[U,V] = Cov[V,W] = Cov[W,U] = -1/3$.
- For Gaussians possible to bound.

Quantitative Arrow – Real Proof

- In fact the proof work under the weaker condition that $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty.
- The reason is that the strong version of majority is stablest (M-10) says:
- If $\min(I_i(f), I_i(g)) < \delta$ for all i and u and v are majority functions with E[f]=u, E[g] = v then:
- $E[f(X)\ g(Y)] < \lim n\ E[u_n(X)\ v_n(Y)] + \varepsilon(\delta)$ where
- $\varepsilon(\delta) \to 0$ as $\delta \to 0$.

Use of Invariance

- Lemma (Kalai-02):
- $PDX(F) = \frac{1}{4}(1 + E[f(x)g(y)] + E[f(x)h(z)] + E[g(y)h(z)])$
- Pf: Look at $s : \{-1,1\}^3 \rightarrow \{0,1\}$ which is 1 on $(1,1,1)$ and $(-1,-1,-1)$ and 0 elsewhere. Then
- $s(a,b,c) = \frac{1}{4}(1+ab+ac+bc)$. $\blacksquare$

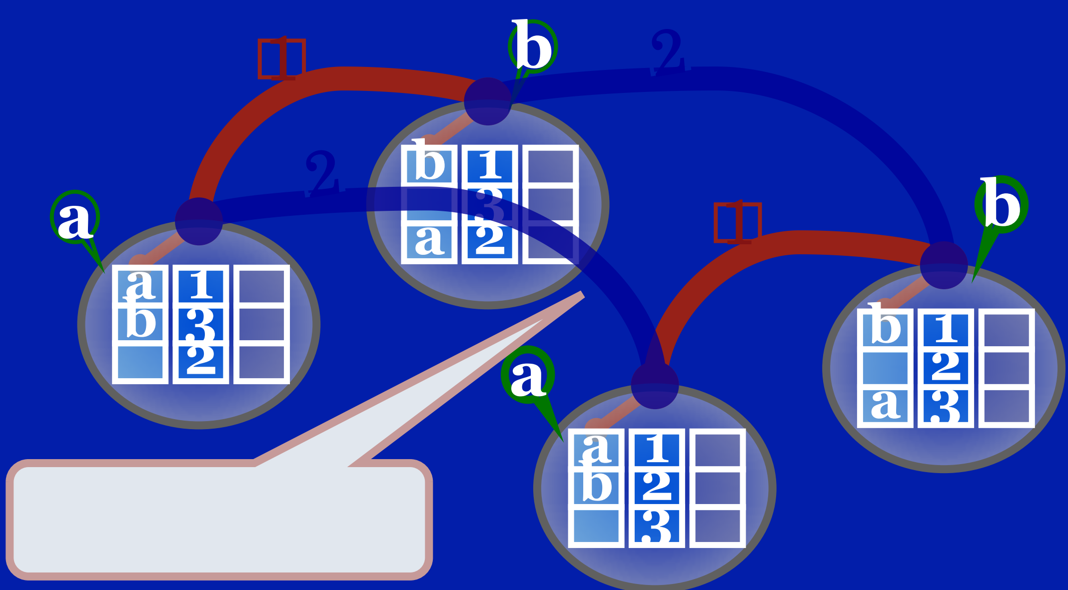
Historical twist

- Condorcet was unhappy with Majority since it may lead to paradox.

- Majority is Stablest implies that majority type functions minimize the probability of a paradox among low influence functions.

A preview of related talk

- A quantitative Gibbard Satterthwaite Thm via
- An isoperimetric result providing lower bounds on the meeting of 3 bodies in the rankings cube
- With Isaksson and Kindler



What is Quantitative Social choice?

- In quantitative social choice – take a second look at these questions.
- Basic question: Is it possible to avoid non-rationality with very good probability?
- Assumes: large number of voters / alternatives.
- In this talk: a quantitative study of Arrow theorem.
- Next talk: study of Gibbard-Satterthwaite thm.

Useful vs. Hard and Work In Progress

- Today:
- Proved impossibility under uniform distribution.
- Bad news.
- Hard proofs.

- Should be done:
- Non uniform distributions.
- Outcomes are rational with high probability.
- Proofs are easy.


Thank you!
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Probability of a Paradox for Low Inf Functions

- Thm: (Follows from MOO-05): $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. If
- $\max_i \max\{I_i(f), I_i(g), I_i(h)\} < \delta$

  then $PDX(F) > \lim_{n \to \infty} PDX(f_n, g_n, h_n) - \epsilon$
- where $f_n = sgn(\sum_{i=1}^n x_i - a_n)$, $g_n = sgn(\sum_{i=1}^n y_i - b_n)$, $h_n = sgn(\sum_{i=1}^n z_i - c_n)$ and a_n, b_n and c_n are chosen so that $E[f_n] \sim E[f]$ etc.
- Thm (Follows from M-08): The same theorem holds

  with $\max_i 2^{nd}(I_i(f), I_i(g), I_i(h)) < \delta$.
- So case I. of quantitative Arrow follows if we can prove Arrow theorem for threshold functions.
- (Recall case I.: $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty)
- Pf for "threshold functions" using Gaussian analysis.

Pf of Majority is Stablest

- Majority is Stablest Conj: If $E[f] = E[g] = 0$ and f,g have all influences less than δ then

  $E[f(x)g(y)] > E[m_n(x)\, m_n(y)] - \epsilon$.
- Ingredients:
- I. Thm (Borell 85): (N_i, M_i) are i.i.d. Gaussians with
- $E[N_i] = E[M_i] = 0$ and $E[N_i\, M_i] = -1/3$, $E[N_i^2] = E[M_i^2] = 1$ and f and g are two functions from R^n to $\{-1,1\}$ with $E[f] = E[g] = 0$ then:
- $E[f(X)\, g(Y)] \geq E[sgn(X_1)\, sgn(Y_1)]$.
- By the CLT: $E[sgn(X_1)\, sgn(Y_1)] = \lim_{n \to \infty} E[m_n(x)\, m_n(y)]$
- II. Invariance Principle [M+O'Donnell+Oleszkiewicz(05)]:
- Gaussian case $\Rightarrow$ Discrete case.

The Geometry Behind Borell's Result

- I. Thm (Borell 85): (N_i, M_i) are i.i.d. Gaussians with
- $E[N_i] = E[M_i] = 0$ and $E[N_i M_i] = -1/3$, $E[N_i^2] = E[M_i^2] = 1$ and f and g are two functions from R^n to $\{-1,1\}$ with $E[f] = E[g] = 0$ then:
- $E[f(X)\, g(Y)] \geq E[sgn(X_1)\, sgn(Y_1)]$.

- Spherical Version: Consider $X \in S^n$ uniform and $Y \in S^n$ chosen uniformly conditioned on $\langle X,Y \rangle \leq -1/3$.
- Among functions f,g with $E[f] = E[g] = 0$ what is the minimum of $E[f(X)\, g(Y)]$?
- Answer: $f = g$ = same half-space.

The Geometry Behind Borell's Result

- More general Thm (Isaksson-M 09): $(N^1,\ldots,N^k)$ are k n-dim Gaussain vectors $N^i \sim N(0,I)$.
- $Cov(N^i,N^j) = \rho\, I$ for $i \neq j$, where $\rho > 0$.
- Then if $f_1,\ldots,f_k$ are functions from R^n to $\{0,1\}$ with E [f] = 0 then:
- $E[f_1(N^1) \ldots f_k(N^k)] \leq E[sgn(N^1_1) \ldots sgn(N^k_1)]$

- Proof is based on re-arrangements inequalities on the sphere.
- Gives that majority maximizes probability of unique winner in Condercet voting for low influence functions.

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