Simulating Constrained Animal Motion Using Stochastic Differential Equations

David R. Brillinger

Department of Statistics
University of California, Berkeley
brill@stat.berkeley.edu

Abstract

Differential equations have long been used to describe the motion of particles. Stochastic differential equations (SDE)s have been employed for situations where randomness is included. This present work is motivated in part by seeking to describe the motion of mammals moving in a constrained region. Interesting questions that arise include: how to write down a pertinent (bivariate) SDE, how to include explanatories, and boundaries and how to simulate realizations of a process?

1 Introduction

Differential equations have long been used to describe the motion of particles and stochastic differential equations (SDE)s have been employed for situations where there is randomness. Our work is motivated in part by the case of ringed-seals, elephant seals, cows, elk and deer. The last three are moving about together in an experimental forest in Oregon.

The study was influenced by emerging data sets in wildlife biology. Biologists and managers wish to use these data sets to address questions such as: how to allocate resources, can different species share a habitat, are changes taking place? One large experiment, Starkey, is described in [6] and [23].

There are technical questions arising of interest to both probabilists and statisticians. Useful tools include: differential equations (DE)s, stochastic differential equations (SDE)s, reflecting stochastic differential equations (RSDE)s, and potential functions

The paper includes review and the results of some elementary simulations, particularly for the case of constrained motion having in mind future data analyses. The work is preparatory to employing simulated realizations of SDE models.

The sections of the paper are: Introduction, Some wildlife examples, Equations of motion, Stochastic differental equations, The constrained case, Results of some simulations, Several particles, and Discussion.

2 Some wildlife examples

The work of the paper, particularly the need to consider bounded domains, may be motivated by some examples from wildlife biology.

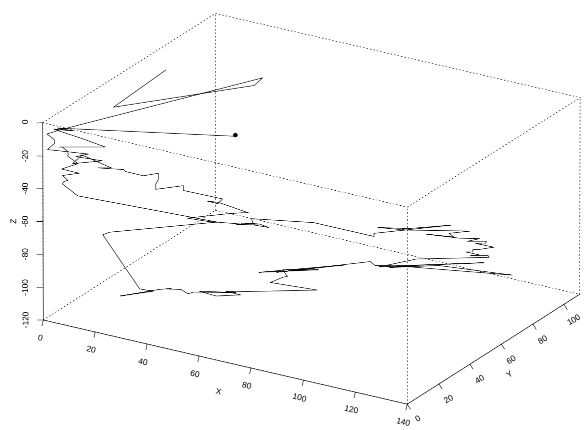

Figure 1: A ringed seal swimming about in an ice-covered lake. The dot indicates the starting location.

Figure 1 shows the motion of a ringed seal as recorded in the Barrow Strait, North West Territories. The animal is constrained within an ice-covered lake that has several air holes. The trip starts at the dot. The animal dives to the bottom, swims around then returns to the air hole. It also looks at another air hole. The locations are available at irregular time intervals. The researchers were concerned with the animal's navigation, foraging and use of its underwater habitat. To study its navigational sense the eyes of the animal were covered during the dive graphed. The ecology of ringed seals is described in [14].

Figure 2 shows the noonday positions of an elephant seal that started out from and returned to an island off Santa Barbara [9]. The dots are the estimated noonday positions. The outward and the inward journeys are shown. Also shown for comparison is a great circle path. The animal's path fluctuates about it. The natural characteristics of the elephant seals are described in [28].

Figure 3 shows the estimated locations of an elk moving about in the Starkey Reserve in Eastern Oregon. There is a high benign fence about

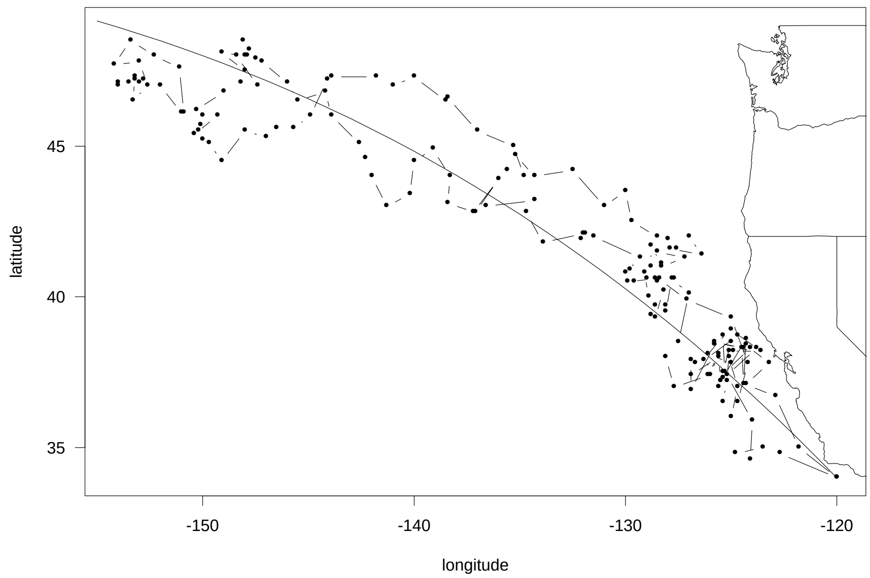

Figure 2: An elephant seal's migration path. The dots are the midday positions.

the reserve. The animal's track is estimated by the curve of broken line segments, the brokedness reflecting the sampling at disparite times. The positions are estimated about every 1.5 hours. The animal keeps moving towards the fence on the southwestern side of Starkey. Details of the experiment may be found in [6].

3 Equations of motion

Differential equations have long been used to describe the motion of particles, see for example [12]. To begin consider one particle moving in the plane. Denote its location at time t by $\mathbf{r}(t) = (x(t), y(t))$. Suppose that there is a potential field, $H(\mathbf{r}, t)$. Such an H controls the direction and speed of the particle. In particular it may be used to describe both attraction and repulsion, for example $H(\mathbf{r}) = |\mathbf{r} - \mathbf{a}|^2$, leads to attraction of the particle to the point $\mathbf{a}$ while $1/|\mathbf{r} - \mathbf{a}|^2$, leads to repulsion from $\mathbf{a}$. Figure 4 includes a perspective plot of an attractive potential in the top left panel.

Nelson, [19], Section 10 discusses the description of such motion. Letting $\mathbf{v}$ denote velocity the equations he sets down are:

$$d\mathbf{r}(t) = \mathbf{v}(t)dt$$

$$d\mathbf{v}(t) = -\beta\mathbf{v}(t)dt - \beta\nabla H(\mathbf{r}(t), t)dt$$

Here ∇ is the gradient $\nabla = (\partial/\partial x, \ \partial/\partial y)$. The quantity $-\beta\nabla H$ is the external force, and β the coefficient of friction. In the case that the friction β is large the equation is approximately

$$d\mathbf{r}(t) = -\nabla H(\mathbf{r}(t), t)dt$$

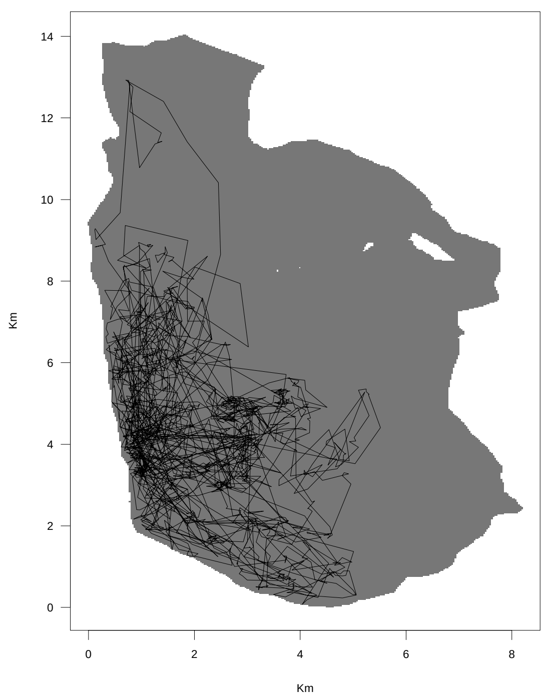

Figure 3: An elk roaming about the fenced Starkey Reserve. The white blobs are fenced-off areas.

This leads to the usual form taken in the study of SDEs, see (2) below.

If H is given, the force field $\mathbf{F}$ is $-\nabla H$. If there exists an H such that $\mathbf{F} = -\nabla H$ then the field $\mathbf{F}$ is called *conservative*. Writing $\mathbf{F} = (F_x, F_y)$ a necessary condition for $\mathbf{F}$ to be conservative is

$$\frac{\partial}{\partial y} F_x = \frac{\partial}{\partial x} F_y \tag{1}$$

If the domain is simply connected, this is also also sufficient and one has

$$H(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

see [29]. The "·" here indicates a line integral.

But, does an H exist? One may use (1) as a check. The Starkey Reserve is not simply connected, see the blobs in Figure 3. Ignoring this, (the scientists said that the fences around the blobs had fallen down), one data analysis, [6], did not rule out the possibility of the existence of a conservative potential field.

4 Stochastic differential equations

Let $\{\mathbf{B}(t)\}$ denote a bivariate Brownian motion. Given the functional parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ consider the equation

$$d\mathbf{r}(t) = \boldsymbol{\mu}(\mathbf{r}(t), t)dt + \boldsymbol{\Sigma}(\mathbf{r}(t), t)d\mathbf{B}(t) \tag{2}$$

Conditions for the existence and uniqueness of solutions may be found in Bhattacharya and Waymire [4], Stroock and Varadhan [30] and Ikeda and Watanabe [13] for example. To tie in with the material of the previous section it may be the case that

$$\boldsymbol{\mu}(\mathbf{r}, t) = -\nabla H(\mathbf{r}, t)$$

for some H.

The motion of $\{\mathbf{r}(t)\}$ may be periodic, for example when there is a seasonal or circadian effect. The motion may be bounded. The parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ may include explanatories, e.g. time of day, distance to nearest road.

4.1 Interpretations

Consider the model (2). Let $H_t = \{\mathbf{r}(u)\text{,t}, \ u \leq t\}$ denote the history of the process up to and including, t, then one has the expressions

$$E\{d\mathbf{r}(t)|H_t\} \approx \boldsymbol{\mu}(\mathbf{r}(t), t)dt$$

$$var\{d\mathbf{r}(t)|H_t\} \approx \boldsymbol{\Sigma}(\mathbf{r}(t), t)dt$$

As well as providing interpretations these relations suggest how $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ might be estimated given data. Examples are developed in [8].

4.2 Solutions and their simulation.

By a solution of the SDE is meant an $\mathbf{r}(t)$ existing given the Brownian process $\{\mathbf{B}(t)\}$, see [4]. Often the way the existence of a solution is demonstrated suggests an algorithm for simulating the process.

Let $\{\hat{\mathbf{r}}(t)\}$ denote an approximation sequence and consider the so-called Euler scheme. It is

$$\hat{\mathbf{r}}(t_{k+1})$$
$$= \hat{\mathbf{r}}(t_k) \; + \; \boldsymbol{\mu}(\hat{\mathbf{r}}(t_k), t_k)(t_{k+1} - t_k) \; + \; \boldsymbol{\Sigma}(\hat{\mathbf{r}}(t_k), t_k)(\mathbf{B}(t_{k+1}) - \mathbf{B}(t_k)) \quad (3)$$

with an initial value $\mathbf{r}(t_0)$, a discretization $\{t_k\}$ of the interval and $k = 0, 1, 2, ...$ Perhaps the t_k will be equi-spaced. The points may be connected by line segments. This and other schemes are investigated in [15].

Next we consider the case where the motion of the particle is constrained.

5 The constrained case

The motion of an animal may be restricted to a region. The ringed seal was in a lake, the elephant seal in a layer at the Earth's surface, and the elk's domain had a high fence about it.

In what follows: a domain D will be given, with boundary ∂D. The constraint may be formalized as requiring $\mathbf{r}(t)$ to be in the closure $\bar{D}$ for all t

5.1 An example: diffusion on a sphere

Figure 2 shows the path of an elephant seal. Here the motion is confined to the surface of the Earth, really to a layer at the surface.

The problem may be formalized as follows: suppose that a particle on the sphere is migrating towards a target at an average speed δ and that the particle is subject to Brownian disturbances of variance σ^2.

In the case that $\delta = 0$ this is the so-called spherical Brownian motion that was studied by Perrin [20].

In [5] the following equations were set down letting θ and ϕ be the colatitude and longitude relative to the target with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. With $(U_t, \; V_t)$ 2 dimensional Brownian motion, consider the process

$$d\theta_t \; = \; (-\delta \; + \; \frac{\sigma^2}{2 \; tan \; \theta_t})dt \; + \; \sigma dU_t$$

$$d\phi_t \; = \; \frac{\sigma}{sin \; \theta_t} dV_t$$

Estimates of the parameters, including the variance of measurement noise, are given in [9] for one data set. In the computations the likelihood function is estimated by simulation.

5.2 Some simulation methods

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There are a number of papers developing the existence and properties of vector diffusions in restricted domains and there are a few that develop simulation methods. References are given below.

Sometimes the parametrization does the constraining. A simple univariate example might involve the path being positive. This could be implimented by writing the process as the exponential of an unconstrained process. In the case of the elephant seal the variables employed did the constraining to the surface of a sphere.

To handle the constrained circumstance researchers often write

$$d\mathbf{r}(t) \;=\; \boldsymbol{\mu}(\mathbf{r}(t),t)dt \;+\; \boldsymbol{\Sigma}(\mathbf{r}(t),t)d\mathbf{B}(t) \;-\; d\mathbf{A}(\mathbf{r}(t),t) \qquad (4)$$

where $\mathbf{A}$ is an adapted process of bounded variation that only increases when $\mathbf{r}(t)$ is on the boundary ∂D. Its purpose is to reflect the particle back to the interior of $\bar{D}$.

Looking for a solution to (4) with appropriate conditions is the so-called Skorohod's problem. Various results have been obtained concerning the existence of solutions, references include: [30], [17], [1], [10], [24], [31], and [25].

Simulations are useful for: program checking, likelihood computation, bootstrapping, and estimating H amongst other things. Three methods are described next. These methods are illustrated in Section 6.

A basic point is that one cannot simply use (3) and throw away a point if it goes outside the boundary for doing so would bias against certain types of behavior.

Method 1. Build a sloping steep wall. That is have a potential term H rising rapidly at the boundary ∂D, when moving from the interior. This leads to the SDE

$$d\mathbf{r}(t) = \boldsymbol{\mu}(\mathbf{r}(t),t)dt \;+\; \boldsymbol{\Sigma}(\mathbf{r}(t),t)d\mathbf{B} \;-\; \nabla H(\mathbf{r}(t),t)dt \qquad (5)$$

The time sequence $\{t_k\}$ will be increasing and $\mathbf{B}(t_{k+1})-\mathbf{B}(t_k)$ written $\sqrt{t_{k+1}-t_k}\,\mathbf{Z}_{k+1}$ where the $\mathbf{Z}_k$s and their entries are independent standard normals. As above the approximant to the value at time t_k will be denoted $\hat{\mathbf{r}}(t_k)$. It is seen that Euler's method given at (3) may be used directly to obtain an approximate solution as in,

$$\hat{\mathbf{r}}_\lambda(t_{k+1}) \;=$$
$$\hat{\mathbf{r}}_\lambda(t_k) \;+\; \boldsymbol{\mu}(\hat{\mathbf{r}}_\lambda(t_k),t_k)(t_{k+1}-t_k)$$
$$+\; \boldsymbol{\Sigma}(\hat{\mathbf{r}}_\lambda(t_k),t_k)\sqrt{t_{k+1}-t_k}\,\mathbf{Z}_{k+1} \;-\; \nabla H(\hat{\mathbf{r}}(t_k),t_k)(t_{k+1}-t_k)$$

with e.g. $H(\mathbf{r}) = \alpha d(\mathbf{r},\partial D)^\beta$ for d distance and scalars $\alpha,\ \beta$.

In the description of the next two methods $\Pi_{\bar{D}}$ will denote the projection operator taking an $\mathbf{r}$ to the nearest point of $\bar{D}$.

Method 2. Penalization scheme. With $\lambda \downarrow 0$ let $\beta_\lambda(\mathbf{r}) \;=\; \{\mathbf{r} - \Pi_{\bar{D}}(\mathbf{r})\}/\lambda$. An approximate solution is now generated via,

$$\hat{\mathbf{r}}_\lambda(t_{k+1}) \;=$$
$$\hat{\mathbf{r}}_\lambda(t_k) \;+\; \boldsymbol{\mu}(\hat{\mathbf{r}}_\lambda(t_k),t_k)(t_{k+1}-t_k)$$
$$+\; \boldsymbol{\Sigma}(\hat{\mathbf{r}}_\lambda(t_k),t_k)\sqrt{t_{k+1}-t_k}\,\mathbf{Z}_{k+1} \;-\; \beta_\lambda(\hat{\mathbf{r}}(t_k))(t_{k+1}-t_k)$$

Some points may lie outside of $\bar{D}$, but small λ brings them close.

Method 3. Projection method. In this case the sequence of approximations to the solution is

$$\hat{\mathbf{r}}(t_{k+1}) \;=$$

$$\Pi_{\bar{D}}\left(\hat{\mathbf{r}}(t_k) \;+\; \boldsymbol{\mu}(\hat{\mathbf{r}}(t_k), t_k)(t_{k+1} - t_k) \;+\; \boldsymbol{\Sigma}(\hat{\mathbf{r}}(t_k), t_k)\sqrt{t_{k+1} - t_k}\;\mathbf{Z}_{k+1}\right)$$

These values do lie in $\bar{D}$. The function $\mathbf{A}$ of (4) may be approximated by

$$\sum_{t_k \le t} (\hat{\mathbf{r}}(t_k) - \Pi_{\bar{D}}(\hat{\mathbf{r}}(t_k)))$$

By a special construction for the case of hyperplane boundaries Lépingle [16] gets faster rates of convergence. He remarks that the constructed process might go outside $\bar{D}$ during some interval t_k to t_{k+1} and provides a construction to avoid this.

Some comparisons

From equation (4)

$$d\mathbf{r} \;=\; \boldsymbol{\mu}dt \;+\; \boldsymbol{\Sigma}d\mathbf{B} \;-\; d\mathbf{A}$$

while from (5)

$$d\mathbf{r} \;=\; \boldsymbol{\mu}dt \;+\; \boldsymbol{\Sigma}d\mathbf{B} \;-\; \nabla H dt$$

so one has the connection

$$d\mathbf{A} \;\approx\; \nabla H dt$$

A crucial difference however is that the support of $d\mathbf{A}$ is on the boundary ∂D while the added term ∇H may be nonzero inside $\bar{D}$.

References for specific methods of simulation are: [18], [21], [22], [26]. Asmussen et al. [2] find that the sampling has to be suprisingly fine in the one-dimnsional case if the Euler method is used. They suggest improved schemes.

One can speculate on how the animals behave when they get to the boundary. They may walk along it for a while. They may run at it and bounce back. They may stand there for a while. This relates to the character of the reflections implicit in the simulation method employed. Dupuis and Ishii [10] allow different types of reflections, including oblique. Ikeda and Watanabe [13] allow "sticky" and "non-sticky" behavior at the boundary.

6 Some simulations

To get practical experience, some elementary simulations were carried out. A naive boundary, namely a circle was employed to make obtaining the result of a projection easy.

Figure 4 shows results for the three methods. There were $n = 1000$ equi-spaced time points and in each case the same starting point and random numbers were employed. The potential function, $\boldsymbol{\mu}$, used is shown on the top left panel of the figure. Its functional form is a standard normal density rotated about the origin. The boundary is taken to be a circle of radius 1.

The top right panel shows the result of Method 1. The term added to the potential function to force the particle to remain in D is proportional to

$$1/(1 - \sqrt{(x^2 + y^2)})^{-3}$$

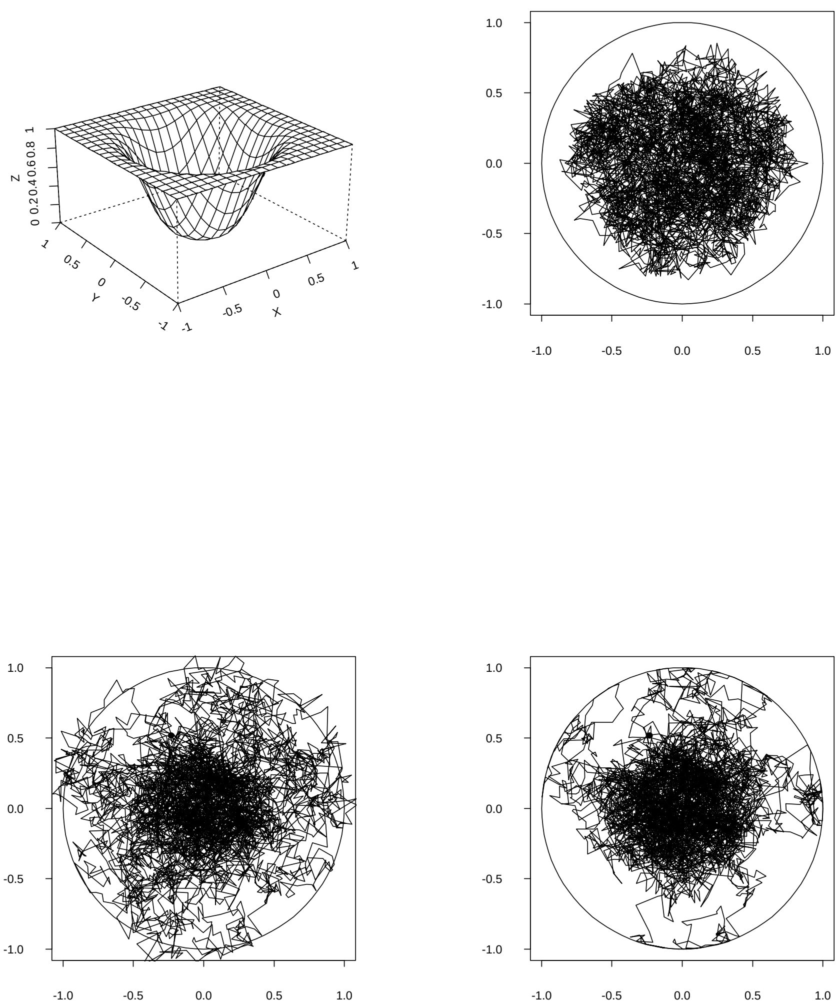

Figure 4: Simulation of a region of attraction at (0,0) and a circular boundary. The top left hand figure is the potential function employed, H. The top right is a simulated trajectory using Method 1. The bottom left used Method 2 and the bottom right Method 3.

This function rises to ∞ on ∂D. The path certainly stays within D and is attracted towards the center. Since the term added is not zero in the region D one is obtaining an approximate solution. When the point moves near the boundary it is repulsed. The bottom left panel shows the result of employing Method 2. The penalization parameter λ was taken to be $t_{k+1} - t_k$. In this case the trajectory goes outside of the circle making the method's approximate nature clear. Of course, by choice of parameters one can make the excursions smaller. The bottom left panel shows the result of employing Method 3, i.e. projection back onto the perimeter of points falling outside. The path stays in the circle.

We learned that the methods were not that hard to program and Method 1 was perhaps the easiest. The running times of the three methods were comparable. Methods 1 and 3 lead to paths in $\bar{D}$. The paths generated by the three methods are surprisingly different despite the random number generator having the same starting point in each case. The presence of the boundary is having an important effect. The path behavior is reminiscent of the sensitivity to initial conditions of certain dynamic systems.

7 Several particles

We begin by mentioning the work of Dyson, [11], [27]. For J particles moving on the line Dyson considered the model

$$dx_j(t) \;=\; \sum_{i \neq j} \frac{1}{x_j(t) - x_i(t)} dt \;+\; \sigma dB_j(t) \tag{6}$$

$j \;=\; 1, 2, ..., J$

This corresponds to the potential function

$$H(\mathbf{x}) \;=\; -\frac{1}{2} \sum_{i \neq j} log(x_j - x_i)^2$$

This function differs from the models considered previously in the paper in being random. One notes that there is long range repulsion amongst the particles and they will not pass each other with probability 1.

Spohn [27] considers the general process

$$dx_j(t) \;=\; -\frac{1}{2} \sum_{i \neq j} H'(x_j(t) - x_i(t))dt \;+\; dB_j(t)$$

where H is a potential function. He develops scaling results and considers correlation functions and Gibbs measures.

Figure 5 presents a simulation of Dyson's process for the case of 2 particles and $\sigma \;=\; .1$. In the figure one sees the particles moving towards 0 repeatedly, but consequently being repelled from each other.

Consider next a more general formulation. Consider particles moving in the plane. Suppose there are J particles with motions described by $\{\mathbf{r}_j(t)\},\; j = 1, ..., J$. Collect the locations at time t into a 2 by J matrix, $\mathbf{s}(t) \;=\; [\mathbf{r}_j(t)]$ and set down the system of equations

$$d\mathbf{r}_j(t) \;=\; \boldsymbol{\mu}_j(\mathbf{s}(t), t)dt \;+\; \boldsymbol{\Sigma}_j(\mathbf{s}(t), t)d\mathbf{B}_j(t), \qquad j = 1, 2, ... \tag{7}$$

with the $\mathbf{B}_j$ independent bivariate Brownians. The Dyson model (6) is a particular case, with special properties.

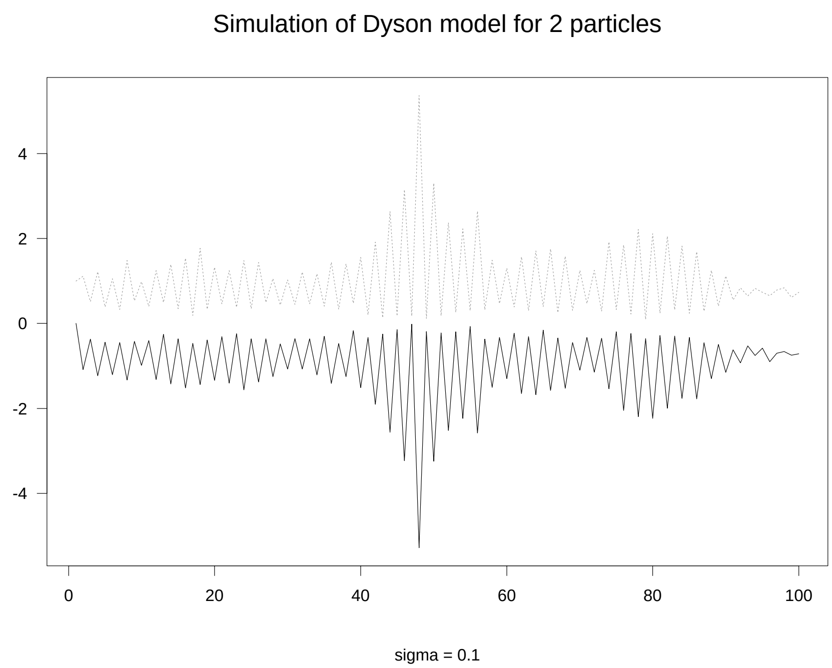

Figure 5: A simulation of Dyson's model (6).

The components may all be required to stay in the same region $\bar{D}$. Questions of interest, e.g. the interactions, now become questions concerning the entries of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Attraction and repulsion might be modelled, e.g. attraction of the animals i and j via setting

$$\boldsymbol{\mu}_{ij}(\mathbf{s}(t)) \;=\; -\nabla\, |\mathbf{r}_i(t) - \mathbf{r}_j(t)|^2$$

One may be able to express the strengths of connection. One might study the properties of the distance $|\mathbf{r}_i(t) - \mathbf{r}_j(t)|$ to learn about the dependence properties amongst the particles. The $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ might include distance to the nearest other particle. There are phenomena to include - animals lagging, clumps, repulsion, attraction, staying about the same distance, ... Lastly there may be animals of several types.

The simulation methods already discussed may be employed here. With data, parameters may be estimated and inferences drawn, e.g. one can study differences of animal behavior. It does need to be remembered that behavior may appear similar because both particles are moving under the influence of the same explanatories rather than inherently connected as in the model (6).

8 Discussion

The paper is principally a review in preparation for statistical work to come. SDEs are the continuing element in the paper. They provide a foundation for the work in particular they offer processes in continuous time, there is an extensive literature, and they have been studied by both probabilists and statisticians.

To a substantial extent the concern of the paper has been with the effect of boundaries. It turns out that there are a several methods for (approximately) simulating processes that are constrained. A small simulation study was carried out to assess relative merits.

Certain practical difficulties arise. These include: choice of sampling times, choice of parameter values, goodness of approximation, the possible presence of lags in a natural model, and the finding of functional forms with which to include explanatories

The regularity conditions have not been laid out. They may be found in the references provided. One can argue that the results are still far from best possible for there is a steady changing of assumptions e.g. re boundedness, convexity and closure.

Many problems remain. There has been some discussion of the case of interacting animals here and in [7]. This is a situation of current concern. In practice it seems that often the process can be only approximately Markov for once the animal has finished some activity it seems unlikely to start it again immediately, e.g. drinking. This means one would like equations including time lags. It is easy to set down such equations, but not so easy to get at the properties of the motion. As an example one might consider

$$dr_1(t) \ = \ \mu_1(r_2(t-\tau))dt \ + \ noise$$

for some function μ_1 and lag τ. The deer may be following the elk at a distance. There are analytic questions such as the expected speeds. There is some literature going under the key words "stochastic delay equation" see [3].

Other interesting questions include:

1. Given the diffusion process (2), how does one tell from the form of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ if there is a closed boundary that keeps the process inside once it starts inside?

One could check to see if $\boldsymbol{\Sigma}(\mathbf{r}(t), t)$ vanishes on ∂D and that $\boldsymbol{\mu}(\mathbf{r}(t), t)$ does not point outside there.

2. How does one include in the model the possibility that the process may follow along the boundary for a period? What are other important types of boundary behavior?

Ikeda and Watanabe's sticky and non-sticky behavior has already been mentioned.

The focus has been on diffusion processes but Lévy processes, with their jump possibilities, seem a pertinent model for some situations. Work does not appear to have been done on the Skorhod problem for Lévy processes.

We have taken an analytic approach in the work and in particular have left for later questions of statistical inference. The tools of model and simulation are basic in the paper and are needed when one turns to the inference issues. Simulation was used to estimate the invariant distribution of the elk in [6] and the likelihood function of an elephant seals's journey in [9].

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