Thematic Semester: Biodiversity and Evolution

Aisenstadt Chair: David J. Aldous

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David Aldous received his Ph.D. at Cambridge in 1977, *and joined the faculty at the University of California Berkeley in* 1979. *His many awards and honours include: the Rollo Davidson Prize in* 1980, *the Loève Prize in* 1993, *a Fellow of the Royal Society in* 1994, *a Fellow of the American Academy of Arts and Sciences in* 2004, *a Fellow of the American Mathematical Society in* 2012. *He was a plenary speaker at the International Congress of Mathematicians in Hyderabad in* 2010.

David J. Aldous

David Aldous is well known for his research on mathematical probability theory and its applications, in particular on topics such as exchangeability, weak convergence, Markov chain mixing times, the continuum random tree and stochastic coalescence. Among many other areas, his work helped develop important stochastic tools for the quantitative analysis of population genetics and evolutionary biology. He held an Aisenstadt Chair in August 2013 during the thematic semester on Biodiversity and Evolution (in the context of the year of Mathematics of Planet Earth 2013).

While his contributions have fundamentally broadened the scope of probability theory, his work was frequently inspired by its use and interpretation in a diverse range of applications. In addition, he continues to write reviews (around 100 so far) of non-technical books involving probability, contributes essays for the *Bernoulli News* and teaches exploratory project-based "lab courses" to undergraduates. Accordingly, in his first lecture (aimed at a broad public audience) he critically discussed instances involving probability in everyday life:

What does mathematical probability tell us about the real world?

Two different approaches were presented:

I. *Fiction*: one can consider what mathematical probability purports to be relevant. Most probability models in undergraduate textbooks are made up in the sense that they make a set of assumptions, which lead to exact formulas, and which have some aspect that a non-mathematician might care about. These models are easy to teach. However, in teaching them one does not commonly attempt to analyze the realism of the model.

Fact: one can also consider theoretical predictions involving randomness in the real world that are verifiable or falsifiable by an undergraduate student (not an experimental scientist) as a course project [7]. Here are some concrete examples:

(1) The Birthday paradox says that in a room of 23 people there is $\approx 50\%$ chance that some two people have the same birthday $\mapsto$ look up a sample of baseball teams of approximately this size.

(2) The regression effect says that if a variable is extreme on its first measurement then it will tend to be closer to its average on its second measurement $\mapsto$ look up performance of the top few best and the bottom few worst teams sports teams in the course of two seasons.

(3) The three arcsine laws for a random walk or Brownian motion say that the following quantities all have the distribution $P(X \le x) = (2/\pi)\arcsin(\sqrt{x})$, $x \in (0,1)$: (i) the proportion of time in a unit time interval that the process is positive, (ii) the last time within a unit time interval when the process changes sign, and (iii) the time at which the process achieves its maximum in a unit time interval $\mapsto$ consider non-overlapping time blocks of short periods of closing values of intra-day stock prices (relies on the usual random walk or Brownian motion models for stock market prices).

(4) The optional sampling theorem for martingales says that if the initial value of a martingale is x, then the probability of reaching or exceeding 100 before it hits 0 is equal to $x/100 \mapsto$ look at prediction betting markets for political elections, where the maximum betting price for each candidate is 100, choose a value x that is larger than all of the candidates' initial prices and count the number of candidates for whom the betting price ever reaches or exceeds x [2].

(5) The Kelly criterion for the borderline between aggressive and insane investing says that given a range of possible portfolios containing risky and safe investments, where a portfolio with allocation strategy α will produce a random return X_α, the optimal long term growth rate is insured by choosing α that maximizes $E\log(1 + X_\alpha)$.

One can imagine taking a number of other topics and trying to find concrete probabilistic statements that can be examined vis-a-vis real data, but evidently this is not as easy as it seems [1].

II. *Perception*: one can alternatively consider in what aspects of life one perceives that chance plays a role, and then examine whether mathematical probability has anything to say that is useful. The more standard directions of academic study deal with philosophy, logic and basic mathematics of probability ("how one should evaluate probabilities"), and with psychology of chance ("how humans, often irrationally, think about chance"). However, these do not quite illustrate how 'ordinary people' think about chance in everyday life. After examining various kinds of data for references to chance (queries to search engines, appearances in blogs and other written material) one can compile an annotated list of contexts in which people, who are not professionally (or in other serious ways) prompted to think about chance, perceive instances of chance. The entries on this list generally fall under events from the past and in the present that one deems unlikely, speculations about the future, and phenomena that we don't usually pay attention to but that can be explained by chance. What is interesting, and emphasized in the lecture, is the disconnect between examples from a typical textbook or other academic inquiry and the examples of chance gathered from everyday life contexts. Aldous' quotes here Nassim Taleb from his book The Black Swan: "*the sterilized randomness of games does not resemble randomness in real life.*"

Many examples of collected real world data, a list of topics that can be used (and have previously been used) in a project-based course at UC Berkeley, the annotated list of a 100 contexts (with illustrative examples) in which we perceive chance, a draft of a book on the subject [4], and other relevant links are available on Aldous' website [5].

In recent years Aldous' theoretical research focused on finite Markov information exchange processes, discrete spatial networks and flows through random networks. His second lecture presented a new take on an area of applied probability that has an extremely broad-ranging use in other scientific fields [3,6].

Interacting particle systems as stochastic social dynamics

Models of individual interactions subject to randomness have been used in physics, computer science and electrical engineering, economics and finance, psychology and sociology, epidemiology and ecology. They are toy models of the ways in which individual 'agents' affect each other, and their goal is to assess the collective result of these interactions on the behaviour of the system as a whole. In mathematics these models are called 'interacting particle systems'; in computer simulations they are referred to as 'stochastic agent-based models.' All of these models are specified by: (i) a graph (or contact network) whose vertices represent agents (or particles, or individuals) and whose edges define the possible pairwise relations between them; (ii) rates (or frequencies) at which information is exchanged between each pair of (neighbouring) agents; (iii) the type of information exchange (or the rule for the state change) that is a result of each interaction. Stochasticity in the model emerges from occurrences of the pairwise interactions: times of all interactions are random and follow a set of Poisson processes with rates prescribed by (i) and (ii); the information exchange at each interaction is also potentially random as specified by (iii). The graph in (i) is often referred to as the underlying geometry of the system, while the rule in (iii) is referred to as the dynamics of the process. The rates in (ii) are often subsumed in the description of the dynamics.

A new technical framework for such models can be formalized as follows: consider a set of agents A and a nonnegative array $\mathcal{N} = (\nu_{ij}, i \neq j \in \mathrm{A} \times \mathrm{A})$ of unordered pairs ($\nu_{ij} = \nu_{ji}$) which is irreducible (the graph of edges on A corresponding to $\nu_{ij} > 0$ entries is connected); assume that each unordered pair $i \neq j$ of agents meets at times given by a rate ν_{ij} Poisson process, independent over different pairs. Let $\mathfrak{S}$ denote a set of possible states for each agent, and $X(t) = \big(X_i(t)\big)_{i \in \mathrm{A}}$ be a stochastic process taking values in $\mathfrak{S}^{\mathrm{A}}$, in which $X_i(t)$ records the state of agent i at time t. The value of $X_i(t)$ changes only at times t at which agent i interacts with some other agent j with $\nu_{ij} > 0$, and is specified by a (deterministic or random) function $F \colon \mathfrak{S} \times \mathfrak{S} \mapsto \mathfrak{S}$ in such a way that if $\big(X_i(t^-), X_j(t^-)\big) = (s_i, s_j)$ then $\big(X_i(t), X_j(t)\big) = \big(F(s_i, s_j), F(s_j, s_i)\big)$. The times at which pairs of agents interact are fully defined by the set of Poisson processes and are called the 'Meeting' process, while the whole process of state changes is referred to as the 'Finite Markov Information Exchange' (FMIE) process.

Many models that have been rigorously analyzed are contained in this description. One of the earliest ones is the classical epidemic model of susceptible, infected and recovered states (SIR) with exponentially distributed waiting times. Of the well known ones from statistical physics the voter model and the contact process (after the usual graph is enriched with one special agent) can be framed as a FMIE. For models in which the population is homogeneously mixing, the underlying graph is the complete graph on the number of agents n with $\nu_{ij} = 1/(n-1)$, $\forall i \neq j \in \mathrm{A}$. For models with Euclidean spatial structure, the finite graph is the discrete d-dimensional torus of strip size $m = \sqrt[d]{n}$ with $\nu_{ij} = 1/(2d)$, $\forall i \sim j$ adjacent in $(\mathbb{Z}/m\mathbb{Z})^d$. In the last two decades more general geometries have been used for the graph of agents, particularly instances of different types of random graphs (Erdős–Rényi, small world, preferential attachment, configuration model with prescribed degree distribution, etc). Graphs with an infinite number of agents can been considered as well (the infinite d-dimensional lattice, or the infinite d-regular tree, etc.), but the focus of this talk was on results that can be made about an FMIE on a finite graph as the number of agents n grows to infinity. The emphasis of the FMIE framework is to view these models as 'stochastic information flow through a network,' and to uncover quantitative aspects of the behaviour of an FMIE model in finite time (rather than in the limit as $t \to \infty$) and specifically their dependence on the geometry of the underlying network.

The following general principles can be useful in applying Markov chain theory to FMIE models: (1) if agents have only a finite number of states it is easy to see what happens in the limit as time goes to infinity; (2) notions of duality and time-reversal are a useful tool in studying the dynamics; (3) bottleneck statistics give crude general bounds; (4) it is often useful to consider some natural coupling of two FMIE processes; (5) certain special families of geometries have local weak limits which are infinite rooted random networks. Here are a couple of examples presented in the talk that illustrate the type of results that can be obtained for FMIE processes, and the power of these general principles.

The '*Pothead*' (or the '*Voter*') model: the state of each agent is one 'opinion' from the set $A = \{1, \ldots, n\}$. Initially each agent i has his own opinion i, when agent i meets agent j we choose uniformly one of the two directions $i \mapsto j$, $j \mapsto i$ and if the direction is $i \mapsto j$ then agent j adopts whatever opinion agent i holds at that time. Let $\mathcal{V}_i(t)$ be the set of agents $\subset$ A who hold the opinion i at time t, then $\mathcal{V}(t) = \left(\mathcal{V}_i(t)\right)_{i\in A}$ forms a random partition of A. Principle (1) tells us that in the long time limit $\mathcal{V}(t)$ will absorb in one of the n possible configurations, in which for some i $\mathcal{V}_i(t) = A, \mathcal{V}_j(t) = \varnothing \ \forall j \neq i$. On can then examine this time to absorption, called the 'consensus time.' Principle (2) can be used to consider the following '*Coalescing*' Markov chain model: initially each agent has one token labelled with his location $\{1, \ldots, n\}$; when agent i meets agent j in the direction $i \mapsto j$ then agent i gives all the tokens he holds to agent j. If we let $\mathcal{C}_i(t)$ be the set of labelled tokens agent i holds at time t, then $\mathcal{C}(t) = \left(\mathcal{C}_i(t)\right)_{i\in A}$ is also a random partition of A, and at any fixed time t the law of $\mathcal{V}(t)$ is equal to that of $\mathcal{C}(t)$. In particular the law of the consensus time is equal to that of coalescence time, which is the first time at which all the tokens in A are held by a single agent. One can easily show that on the complete graph for agents, the expected coalescence time is approximately $2n$. Principle (3) can then be used to estimate the expected consensus time on a graph with general geometry using bottleneck statistics on weighted graphs (such as the isoperimetric constant).

The '*Pandemic*' (and the '*First Passage Percolation*') model: the state of each agent is $\{0, 1\}$ recording whether the agent is infected or not; initially only one agent is infected, when an infected agent meets another the other becomes infected as well. The inverse of the rates of meetings define a notion of distance on the underlying graphs, and in the special case when all the strictly positive ν_{ij} are equal to 1 this is equivalent to the dynamical version of the '*First Passage Percolation*' process with Exponentially distributed passage weights. The pandemic model has been studied on many geometries, and exhibits the fastest possible spread of information on any FMIE model. On the complete graph for n agents one can obtain the following limit result for the proportion of infected agents $X_n(t)$: there exists a sequence of random variables G_n such that $\sup_t |X_n(t) - F(t - \log n - G_n)| \to 0$ in probability, where $F(t)$ is the logistic function, $\log n$ is the length of the initial phase of the infection, and G_n converges in law to a Gumbel distribution. Principle (4) allows one to apply these results to a number of other models that can be built upon the Pandemic model. Principle (5) can be used to get estimates for the expected spread of the epidemic, or the expected time until the infection spreads from agent i to a prespecified agent j, on a d-dimensional torus from known shape theorems for First Passage Percolation on the infinite d-dimensional lattice. Analogous estimates for general geometries remain unsolved.

In his third lecture Aldous described a specific Finite Markov Information Exchange process he recently proposed whose analysis gives rise to some interesting new mathematical objects.

The Compulsive Gambler process and the Metric Coalescent

Consider the following FMIE model called The 'Compulsive Gambler' process: the state of each agent is the amount of money he has, initially all agents have an equal amount of money, when two agents meet they play a fair winner-takes-all gamble (that is, the chance for each agent to win is proportional to the amount of money he brings to the gamble), and whichever agent wins takes all of the money from the agent that lost. Notice that once an agent loses a single gamble, he will have no chance of winning any other gamble in the future. Let $X_i(t)$ denote the amount of money agent i has at time t. In a model with finitely many agents, n, the total amount of money all the agents have is constant in time.

This process is interesting from a methodological point of view. The following techniques were useful in analyzing its behaviour: (1) martingales; (2) comparison with the Kingman coalescent chain; (3) ordered version of the model; (4) exchangeability.

(1) One can immediately observe that the number of agents which hold non-zero amounts at time t can only decrease in time, and the system will absorb as soon as one of the agents has all the wealth, when $X_i(t) = 1$ for some i. In case the geometry of the weighted graph of agents satisfies $\nu_\star = \min_{i,j} \nu_{ij} > 0$, then a simple martingale argument shows that in the long time limit the chance for each agent to accumulate all the money is proportional to his initial wealth. The collection of martingales $M_f(t) = (1/n) \sum_i f(i) X_i(t)$ for any function f on agents, whose second moment is bounded by $\nu^\star t$, can be used to show weak convergence of the empirical measure of the money allocation process.

(2) If the underlying network is a complete graph $\nu_{ij} = 1, \forall i \neq j$ then the number of agents with non-zero amounts is distributed as a Kingman coalescent chain. Using $\nu_\star = \min_{i,j} \nu_{ij}$ and $\nu^\star = \max_{i,j} \nu_{ij}$, one can estimate the time of wealth concentration T in a general graph by simple comparisons with the coalescence time of the Kingman chain, $2(1 - 1/n)/\nu^\star \leq ET \leq 2(1 - 1/n)/\nu_\star$.

(3) An ordered version of the model is as follows: suppose each unit of money is initially assigned an i.i.d. random label, and consider the ordered version of the gamble in which when two agents meet, the winner is determined as the owner of the unit with the lowest serial number. Notice that the owner of the unit that has the lowest serial number of all will eventually accrue all of the money. This process has the same distribution as the Compulsive Gambler process.

(4) The exchangeability property of this model refers to the conditional law of the money allocation process: given the amount of money each agent has, the ownership of serial numbers among agents is uniformly distributed on the set of all compatible partitions of A. As an immediate consequence, the agent who ultimately acquires all of the money is uniform random on A.

A new process, called the '*Metric Coalescent*,' was inspired by considering an abstract extension of this process: assume the total initial wealth is 1 and consider it as an arbitrary probability measure on A; instead of placing agents on the vertices of a graph consider them as points $a_1, \ldots, a_n$ in a metric space $(\mathcal{A}, d)$; define meeting rates as functions of their distance $\nu_{ij} = \phi\big(d(a_i, a_j)\big)$; construct an empirical measure $\mu(t) = \sum_i X_i(t)\delta_{\xi_i}$ where ξ_i are i.i.d. locations of agents distributed as μ. The Compulsive Gambler process is equivalent to the evolution of this empirical measure in time. It is very likely that one can show that for each μ on $\mathcal{A}$ there exists a unique probability measure valued process $\mu(t)$ which evolves as the Compulsive Gambler from any time $t_0 > 0$ onwards and whose a.s. limit as $t \downarrow 0$ is equal to μ. The methodologies outlined above can be particularly useful in proving this result.

Aldous' lectures displayed an arsenal of classical tools in probability theory. The new FMIE framework he described proved the power of these tools to create new paradigms, when used creatively.

[1] D. J. Aldous, *The great filter, branching histories, and unlikely events*, Math. Sci. **37** (2012), no. 1, 55–64.

[2] ———, *Using prediction market data to illustrate undergraduate probability*, Amer. Math. Monthly **120** (2013), no. 7, 583–593.

[3] ———, *Interacting particle systems as stochastic social dynamics*, Bernoulli **19** (2013), no. 4, 1122–1149.

[4] ———, *On chance and unpredictability*: 13/20 *lectures on the links between mathematical probability and the real world*, available at http://www.stat.berkeley.edu/~aldous/Real-World/draft_book.pdf.

[5] ———, *Overview of probability in the real world project*, available at http://www.stat.berkeley.edu/~aldous/Real-World/cover.html.

[6] D. J. Aldous and D. Lanoue, *A lecture on the averaging process*, Probab. Surv. **9** (2012), 90–102.

[7] D. J. Aldous and T. Phan, *When can one test an explanation? Compare and contrast Benford's law and the fuzzy CLT*, Amer. Statist. **64** (2010), no. 3, 221–227.

Martin Nowak

(*continued from page 4*)

Kin selection occurs among genetically related individuals: "*I will jump into the river to save two brothers or eight cousins.*" (J.B.S. Haldane) The evolution of cooperation is then related to Hamilton's rule which states that it can occur if $r > c/b$, where r is a probability of sharing a gene which measures relatedness.

Direct and indirect reciprocity are essential for understanding the evolution of any pro-social behaviour in humans. Citing Martin Nowak: "But 'what made us human' is indirect reciprocity, because it selected for social intelligence and human language."

Martin Novak ended his fascinating Grande Conférence publique with an image of the Earth and the following sentence: "*We must learn global cooperation… and cooperation for future generations.*" This started a passionate period of questions, first in the lecture room, and then around a glass of wine during the vin d'honneur.

In his previous talk opening the workshop, "Evolution of sociality", Martin Nowak had compared two reproductive scenarios to make simple and testable predictions: a solitary life style with all offspring leaving to reproduce, and an eusocial life style with some offspring staying and helping raise further offspring. He had also shown the limitations of an inclusive fitness maximization approach that consists in making predictions in the presence of interactions between individuals by transferring fitness effects from recipients to actors weighted by coefficients of relatedness.

In his more technical talk delivered later on in the workshop, Martin Nowak presented an overview of stochastic models of "Evolutionary dynamics" in well-mixed populations as well as structured populations, including populations on graphs and on sets and extensions to n-strategy games.

Number Theory from Arithmetic Statistics to Zeta Elements

(*continued from page 2*)

there isn't a single lab member who isn't involved in organizing at least one workshop. The very first workshop is being organized by Chantal David…

AG: *together with an international committee* [Pär Kurlberg, Zeév Rudnick]*…*

HD: *and then the November workshop is organized by Andrew and me…*

AG: *I am organizing the second one in additive combinatorics* [with David Conlon, Ben Green, Laurent Habsieger, Alain Plagne], *the third one is the two of us* [with Jordan Ellenberg], *plus Dick Gross at Harvard* [co-organizer of the SMS Summer school], *and then the fourth one is this exciting thing for young people…*

HD: *organized by Addario-Berry, who is actually not a part of our number theory group…*

AG: *and Dimitris Koukoulopoulos* [from CICMA].

HD: *The second semester will be taken up by more arithmetic activities. The first workshop* [Regulators, Mahler measures, and special values of L-functions, February 16–20, 2015] *is being run by Matilde Lalín and me* [with Wadim Zudilin]. *Matilde is a mathematician whose expertise straddles both the analytic and the algebraic aspects of the subject. Her interests have been broadening towards the analytic direction recently, but a central focus is still the special values of L-functions and their interpretation in terms of the conjectures of Beilinson–Bloch, so we really think there is an occasion there to share ideas across boundaries.*

HD: *The next workshop is the one on p-adic methods which I'm organizing with Adrian Iovita, Matt Greenberg from Calgary, and Payman Kassaei, who is one of our newest members. Finally, the third workshop in the second semester is being organized by Eyal Goren and me, and will focus on the Kudla program.*

Bulletin: And there will be some courses too?

HD: *Exactly. The graduate courses have not completely been determined for the second semester, but we certainly expect that at least two of Eyal Goren, Adrian Iovita, Payman Kassaei and me will be teaching a graduate course aimed at the beginning and intermediate level graduate student.*