Background: study of the **ground states of disordered systems** (statistical physics) is related to **probabilistic analysis of optimization problems** (computer science).

Define prob. dist. on configurations $\mathbf{x}$ by

$$\pi_T(\mathbf{x}) = K_T \exp(-H(\mathbf{x})/T).$$

For $T = 0$ the dist. lives on the ground states where $H(\mathbf{x})$ is minimized.

ordered system: $H(\mathbf{x})$ is simple function of $\mathbf{x}$ and a few numerical parameters.

disordered system: $H = H(\mathbf{x}, \theta)$ for some random environment θ.

3 different angles on "ground states of a disordered system".
(C.S.) Study average time-complexity of some algorithm for finding $\mathbf{x}(\theta) := \arg\min_{\mathbf{x}} H(\mathbf{x}, \theta)$.
(Math.) Study dist. of $H^*(\theta) = \min_{\mathbf{x}} H(\mathbf{x}, \theta)$.
(Stat. Phys.) Study structure of typical optimal configurations $\mathbf{x}(\theta)$.

The analogy has been familiar since early 1980s (**simulated annealing** and all that). But in contrast to e.g. **Markov chain sampling algorithms** there are very few actual theorems around the interface (e.g. **random 2-SAT**).

More background: **uniqueness** issues and the **weak convergence paradigm**.

Consider the TSP (traveling salesman problem) on n random points in $[0,1]^2$. Optimal tour T_n has length $L(T_n)$ satisfying $L(T_n) \sim c_{\mathrm{BHH}}\ n^{1/2}$ for constant c_{BHH}. For a non-optimal tour T'_n define

$$\Delta(T'_n) = \text{number of edges in } T'_n \setminus T_n.$$

Fix $\varepsilon > 0$ and consider the set of "near-optimal" tours T'_n with $L(T'_n) \leq (c_{\mathrm{BHH}} + \varepsilon)n^{1/2}$. Obviously one can create a near-optimal tour by altering the optimal tour on some small proportion of edges. But is this the <u>only</u> way to get near-optimal tours? More precisely, does the random TSP have the following property?

The AEU (asymptotic essential uniqueness) property?

$\forall \varepsilon > 0 \ \exists \delta(\varepsilon)$ such that, with prob $\to 1$ as $n \to \infty$,

$\forall T'_n : \ \dfrac{L(T'_n)}{n^{1/2}} \leq (c_{\mathrm{BHH}} + \varepsilon)$ implies $\dfrac{\Delta(T'_n)}{n} \leq \delta(\varepsilon)$.

(require $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$).

We can define the AEU property similarly for many other random optimization problems.

For the TSP it seems natural to conjecture that the AEU property fails; also conjecture this is hard to prove!

The **weak convergence paradigm**: sometimes the $n \to \infty$ limit of finite problems is a certain "infinite" problem. Here's how it works for the Euclidean TSP.

Poisson (rate 1) point process on R^2. Collect points into infinite paths in some arbitrary (but translation-invariant) way, so by ergodicity we can define

$$(*) \qquad E(\text{ length of typical edge}).$$

Then c_{BHH} is the minimum of (*) over all path-collections. The minimum is attained, by compactness. The AEU property (?) is the property that the minimum path-collection is a.s. unique.

In TSP setting, this reformulation isn't helpful. But it <u>is</u> helpful for the problem we shall study.

An elementary probability fact.

Take n independent r.v.'s with the exponential (mean n) distribution. Put them in increasing order as $0 < \xi_1^{(n)} < \xi_2^{(n)} < \ldots$. Then as $n \to \infty$

$$(\xi_1^{(n)}, \xi_2^{(n)}, \ldots) \ \overset{d}{\to} \ (\xi_1, \xi_2, \ldots)$$

where limit is the Poisson(1) point process.

In particular, for fixed k we have $E\xi_k^{(n)} \to k$.

The **assignment** (= **bipartite matching**) problem is a special case of LP. Imagine n tasks, n machines, cost y_{ij} to do task i on machine j, and only one task allowed per machine. Seek to minimize $\sum_{i=1}^{n} y_{i,\pi(i)}$ over permutations π of $\{1, 2, \ldots, n\}$.

Random Assignment (= **random bipartite matching**) problem.

$n \times n$ random matrix Y_{ij} with independent exponential(mean n) entries. Study

$$A_n = n^{-1} \ \min_{\pi} \sum_{i=1}^{n} Y_{i,\pi(i)}$$

the minimum over permutations π of $\{1, 2, \ldots, n\}$. Steele (1997 monograph) gives lengthy discussion. Brief history:
(i) LP methods, e.g. Karp (1987) shows
$EA_n \in [1 + e^{-1}, 2]$ asymptotically.
(ii) Conjectured exact formulas (Parisi $<$ 1998)

$$EA_n = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \ldots + \tfrac{1}{n^2}$$

and generalized conjectures in Coppersmith - Sorkin (1999).
(iii) Mezard - Parisi (1987) studied it as toy model of disordered system; used heuristic replica method to argue

$EA_n \to \zeta(2) = \pi^2/6$ (asymptotic conjecture).

Will outline proof of the asymptotic conjecture. `www.stat.berkeley.edu/`

How does the logistic distribution arise?

Consider the fixed-point problem: what distribution dist(X) satisfies

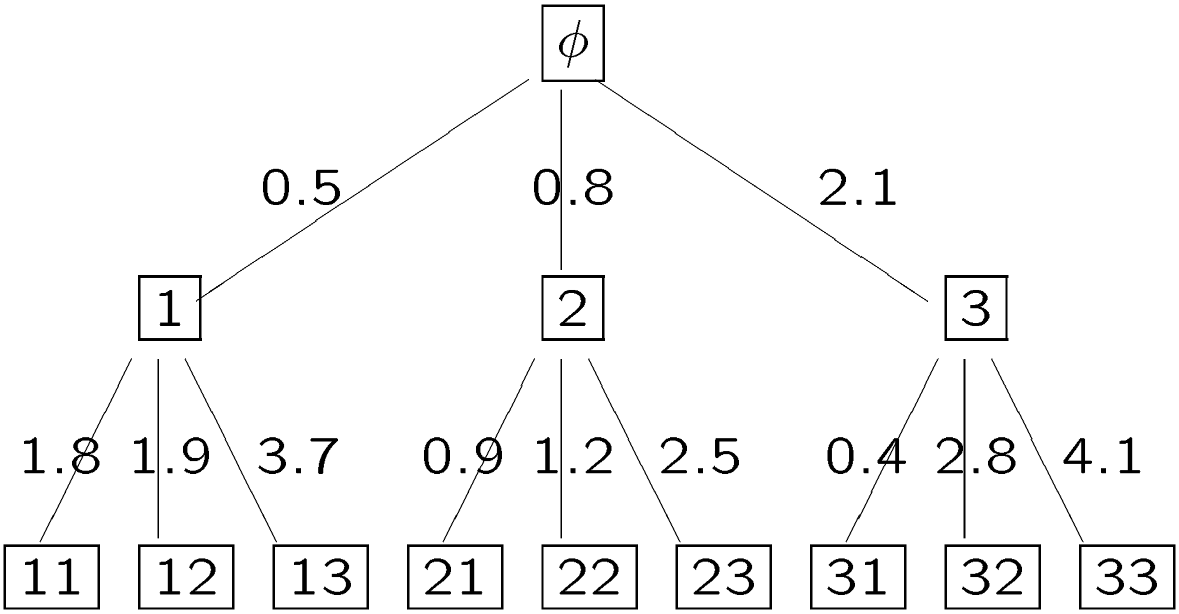
$$X \overset{d}{=} \min_{1 \le i < \infty} (\xi_i - X_i)$$

where the (ξ_i) are the points of a Poisson(1) process?

Such problems occur often in probabilistic analysis of recursive algorithms (e.g. QUICKSORT) and other "probability on trees" contexts. Usually no explicit solutions; here "a miracle occurs" and easy to show the logistic distribution is unique solution.

Amazingly, the above are the only "calculation" ingredients of the proof – the rest is "pure thought". We use the weak convergence paradigm, so first need to describe a limit random structure.

A realization of the PWIT (Poisson-weighted infinite tree). For each vertex v, the weights $W(v, v1)$, $W(v, v2)$, $W(v, v3)\ldots\ldots$ on the downward edges are the points of a Poisson(1) process.



This PWIT is a "local limit" of the $n \times n$ randomly-weighted complete bipartite graph.

The PWIT has a certain "spatial invariance" property, which formalizes the idea

the distribution of edge-weights of the PWIT doesn't depend on which vertex we choose as the root ϕ.

One can then define the notion of a spatially invariant matching $\mathcal{M}$ on the PWIT. Note that the "greedy matching", which starts by matching ϕ with vertex 1, is not spatially invariant. A spatially invariant matching $\mathcal{M}$ has a mean cost

$$c(\mathcal{M}) := EW(\phi, \mathcal{M}(\phi)).$$

Aldous (1992) follows the weak convergence paradigm to show

Theorem 1 $\lim_n EA_n = \inf_{\mathcal{M}} \ c(\mathcal{M})$, *the infimum over all spatially invariant matchings on the PWIT.*

How to define an optimal matching?

Given a realization of the PWIT, consider defining

$$\begin{aligned} X_\phi &= \text{cost of optimal matching on } \mathbf{T} \\ &- \text{cost of optimal matching on } \mathbf{T} \setminus \{\phi\}. \end{aligned}$$

Here we mean *total* cost, so we get $\infty - \infty$, and so this makes no sense as a rigorous definition, though statistical physics uses such renormalization arguments all the time. But pretend it does make sense. Then for each vertex v we can define X_v similarly in terms of the subtree $\mathbf{T}^v$ rooted at v (the vertices of $\mathbf{T}^v$ are v and its descendants):

$$\begin{aligned} X_v &= \text{cost of optimal matching on } \mathbf{T}^v \\ &- \text{cost of optimal matching on } \mathbf{T}^v \setminus \{\phi\}. \end{aligned}$$

And we get the recursion

$$X_v = \min_{1 \le j < \infty} (W(v, vj) - X_{vj}) \qquad (1)$$

because the left side is the cost difference between using or not using v in a matching on $\mathbf{T}^v$; to use an edge (v, vj) we have to pay the cost of the edge and the difference between the cost of not using or using vj, which is the right side. Moreover in the optimal matching on $\mathbf{T}^v$, vertex v should be matched to the vertex vj attaining the minimum in (1).

Now make a rigorous construction. Each edge $e \in \mathbf{E}$ of $\mathbf{T}$ corresponds to two directed edges $\overrightarrow{e}, \overleftarrow{e}$: write $\overrightarrow{\mathbf{E}}$ for the set of directed edges.

Lemma 2 *Jointly with the edge-weights* $(W(e), e \in \mathbf{E})$ *of the PWIT we can construct* $\{X(\overrightarrow{e}), \overrightarrow{e} \in \overrightarrow{\mathbf{E}}\}$ *such that*
(i) each $X(\overrightarrow{e})$ *has the logistic distribution*
(ii) for each $\overrightarrow{e}$*, with children* $\overrightarrow{e}_1, \overrightarrow{e}_2, \ldots$ *say,*

$$X(\overrightarrow{e}) = \min_{1 \le j < \infty} (W(\overrightarrow{e}_j) - X(\overrightarrow{e}_j)) \qquad (2)$$

Guided by the heuristic argument, for each vertex v define

$$v^* = \arg\min_{v' \sim v} (W(v, v') - X(v, v')). \qquad (3)$$

It is easy to check $(v^*)^* = v$. Thus the set of edges (v, v^*) is a matching $\mathcal{M}_{\text{opt}}$ on the PWIT, and it turns out

$$(v, v') \in \mathcal{M}_{\text{opt}} \text{ iff } W(v, v') < X(v, v') + X(v', v).$$

So to complete a proof of the asymptotic conjecture, need to show

(a) the cost $c(\mathcal{M}_{\text{opt}}) = EW(\phi, \mathcal{M}_{\text{opt}}(\phi))$ equals $\pi^2/6$

(b) any other spatially invariant matching $\mathcal{M}$ has $c(\mathcal{M}) \geq c(\mathcal{M}_{\text{opt}})$.

Moreover if we can show strict inequality $c(\mathcal{M}) > c(\mathcal{M}_{\text{opt}})$ then we will establish the AEU property for the random assignment problem.

Intuitively, the quantities $X(\overrightarrow{e})$ must be analogous to *dual variables* in linear programming. But I am unable to make this idea more precise.

Easy to calculate $c(\mathcal{M}_{\mathsf{opt}})$.

$$P(W(\phi, \mathcal{M}_{\mathsf{opt}}(\phi)) \in [w, w + dw]) =$$

chance some edge (ϕ, j) has weight $\in [w, w + dw]$

$$\times\ P(w < X(\phi, j) + X(j, \phi))$$

$$= dw\ \times\ h(w).$$

To study optimality, consider some other spatially invariant matching $\mathcal{M}$, and let Ω be the event that $\mathcal{M}(\phi) \neq \mathcal{M}_{\mathrm{opt}}(\phi)$. A brief calculation gives

$$c(\mathcal{M}) - c(\mathcal{M}_{\mathrm{opt}}) = E 1_{\Omega} \partial(\phi)$$

for

$$\partial(\phi) = (W(\phi, \mathcal{M}(\phi)) - X(\phi, \mathcal{M}(\phi)))$$

$$- \overset{[2]}{\min_{v \sim \phi}} (W(\phi, v) - X(\phi, v))$$

$$\geq 0$$

where $\min^{[2]}$ denotes second-minimum. This establishes weak optimality for $\mathcal{M}_{\mathrm{opt}}$. To study strict optimality, consider the <u>partial</u> matching $\mathcal{M}^{[2]}$ consisting of edges (v, v') such that

$$v' = \arg \overset{[2]}{\min_{y \sim v}} (W(v, y) - X(v, y))$$

$$v = \arg \overset{[2]}{\min_{y \sim v'}} (W(v', y) - X(v', y))$$

For a matching $\mathcal{M}$ to have the same cost as $\mathcal{M}_{\text{opt}}$, each edge of $\mathcal{M}$ must be an edge of $\mathcal{M}_{\text{opt}}$ or $\mathcal{M}^{[2]}$. In other words there must exist infinite paths consisting of alternating edges from $\mathcal{M}_{\text{opt}}$ and $\mathcal{M}^{[2]}$. It is possible to show this can't happen.