Optimal flows through the disordered lattice

Consider lattice Z^2 with i.i.d.r.v.'s (C_e) on edges e. This structure can be used in many settings

- first passage percolation
- RWIRE
- disordered Ising model
- disordered variants of many interacting particle systems.

We'll use it for "traffic flow" models.

Big Picture. Instead of studying only graph structure of networks, think "what does the network do?" One answer: "move stuff from place to place". Envisage road traffic.

Complete info about routes is the **path-flow**, a measure μ on (directed, loop-free) paths

$$\text{source} = v_0, v_1, v_2, \ldots, \text{destination}.$$

Associated with a path-flow is its induced **flow-volume** $\mathbf{f} = (f(e))$,

$$f(e) = \text{ volume of flow across edge } e$$

(both directions combined) and its induced **transportation measure** on (source,destination).

Optimal routing problem: Given

- network
- transportation measure
- cost function depending on $(f(e))$
- capacity constraints $(\text{cap}(e))$

we ask

does there exists a feasible path-flow?

if so, what is minimum-cost feasible flow?

Deterministic algorithmic problems like this are studied as part of **network algorithms**; as multicommodity flow problems they are NP-hard in general. We take **statistical physics viewpoint** of modeling the network (topology, costs, constraints) as random and studying properties of optimal solution. We take transportation measure uniform on all (source,destination) pairs, so there's one parameter

$\rho =$ normalized traffic demand.

Seek to study (in different models on n-vertex networks) the $n \to \infty$ limit curves giving some quantitative measure of network performance vs ρ.

In many random-graph like networks we hope to exploit the "locally tree-like" structure to do explicit calculations. But what about disordered Z^2?

Order-of-magnitude calculation on $N \times N$ grid. Send volume ρ_N between each (source,destination) pair. Then average flow volume $\bar{f}$ across edges has

$$(N^2 \times N^2) \times \rho_N \times N \approx \bar{f} \times N^2$$

We take

$$\rho_N = \rho N^{-3}$$

so that flow-volumes $f(e)$ will be order 1.

Open Problem. Take i.i.d. capacities (cap(e)) with $0 < c_- \leq$ cap(e) $\leq c_+ < \infty$. Then a feasible flow with normalized demand ρ exists for $\rho < \rho_-$ and doesn't exist for $\rho > \rho_+$. Prove there is a constant ρ_* depending on distribution of cap(e) such that as $N \to \infty$

$$P(\exists \text{ feasible flow, norm. demand } \rho) \begin{array}{ll} \to 1 & , \ \rho < \rho_* \\ \to 0 & , \ \rho > \rho_*. \end{array}$$

Instead of focussing on capacities, let's focus on <u>congestion</u>. In a network without congestion, the cost (to system; all users combined) of a flow of volume $f(e)$ scales linearly with $f(e)$. With congestion, an extra user imposes extra costs on other users as well as on themself. So cost scales super-linearly with $f(e)$.

Model: The cost of a flow $\mathbf{f} = (f(e))$ in an environment $\mathbf{c} = (c(e))$ is

$$\text{cost}_{(N)}(\mathbf{f}, \mathbf{c}) = \sum_e c(e) f^2(e).$$

Theorem 1. $N \times N$ <u>torus</u> (for simplicity)
Large constant bound B on edge-capacity (for simplicity)
i.i.d. cost-factors $c(e)$ with

$$0 < c_- \leq c(e) \leq c^+ < \infty.$$

Let Γ_N be minimum cost of flow with normalized intensity $\rho = 1$. Then

$$N^{-2} E \Gamma_N \to \text{constant}(B, \text{dist}(c(e))).$$

Note: Easy concentration-of-measure lemma then implies

$$N^{-2} \Gamma_N \to \text{constant}(B, \text{dist}(c(e)))$$

in probability.

Idea of proof. Consider optimal flow in a randomly positioned $M \times M$ window; consider $N \to \infty$ weak limits.

Note: As $N \to \infty$ the volume of flow with source or destination at a vertex v becomes negligible compared to the flow through v.

Weak limit flows across the $M \times M$ square.

- i.i.d. environment $(c(e))$
- a path-flow across the square
- with a transportation measure Q on the boundary $\text{Bou}_M \times \text{Bou}_M$
- Q is dependent on $(c(e))$
- given $(c(e))$ and Q, the path-flow inside the square minimizes the local cost

$$\text{cost}_M(\mathbf{f}, \mathbf{c}) = \sum_e c(e) f^2(e).$$

Consider Q non-random. So there's an expectation (over random environment)

$\text{cost}_M(Q) := E \inf\{\text{cost}_M(\mathbf{f}, \mathbf{c}) : \mathbf{f} \text{ has t.m. } Q\}$.

Note

- $Q \to \text{cost}_M(Q)$ is convex
- An easy concentration inequality (C.I.) shows that the r.v.

$$\inf\{\text{cost}_M(\mathbf{f}, \mathbf{c}) : \mathbf{f} \text{ has t.m. } Q\}$$

is close to its expectation.

These ingredients suggest the following conceptually standard (but technically hard to implement here) argument. Take a large finite set $\mathcal{Q}$ which is δ-dense in space of Q's.

$N \to \infty$ weak limit involves

$$\text{cost}_M(\mathbf{f}, \mathbf{c}) : \ \mathbf{f} \text{ has random t.m. } Q$$

Use δ-dense to say

$$\approx \text{cost}_M(\mathbf{f}, \mathbf{c}) : \ \mathbf{f} \text{ has random t.m. } Q \in \mathcal{Q}$$

[by C.I.] $\approx$ weighted ave of $\text{cost}_M(Q)$ over $Q \in \mathcal{Q}$

$$[\text{convexity}] \geq \text{cost}_M(EQ) \qquad .$$

Of course we don't know EQ but we'll write some constraints soon. This argument gives a lower bound for Theorem 1

$$\liminf N^{-2} E\Gamma_N \geq -\varepsilon_M$$

$$+M^{-2} \inf\{\text{cost}_M(Q^0) : \ Q^0 \text{ satisfy constraints}\}.$$

What are constraints on $Q^0 = EQ$ arising from Q being t.m. across random $M \times M$ window in a uniform source-destination flow on the $N \times N$ torus?

Recall Q is a measure on pairs $(v_{\text{ent}}, v_{\text{exi}}) \in R^2 \times R^2$. Has marginals Q_{ent} and Q_{exi}.

Constraint 1. The push-forward of Q^0_{exi} under the reflection map equals Q^0_{ent}.

Constraint 2. Write

$$\text{drift}(Q) = M^{-2} \int (v_{\text{exi}} - v_{\text{ent}}) \, dQ.$$

Then Q^0 has a mixture representation

$$Q^0 = \int Q \ \psi(dQ)$$

where ψ is a p.m. whose pushforward under the map

$$Q \to \text{drift}(Q) \bmod (1,1)$$

is uniform on the continuous torus $[0,1)^2$.

Ultimately we prove

$$\lim N^{-2} E \Gamma_N =$$

$$\lim_M M^{-2} \inf\{\text{cost}_M(Q^0) : Q^0 \text{ satisfy constraints}\}.$$

What do we need to do, to prove the upper bound? Fix some Q^0 satisfying constraints. We need to construct a flow on the $N \times N$ torus such that

$$\limsup_N N^{-2} E \text{cost}_{(N)}(\mathbf{f}, \mathbf{c})$$

$$\leq M^{-2} \text{cost}_M(Q^0).$$

Rather magically, our previous abstract [non-constructive] arguments provides clues for the construction. A transportation measure (t.m.) Q can be normalized to a transition matrix (t.m.). So we can use any given Q to define a Markov chain on the "skeleton" of the partition.

Take N/M steps of this chain starting from some vertex v_1. We finish at some random vertex v_2 with

$$v_2 - v_1 \approx N\text{drift}(Q)$$

using Markov chain LLN. So we construct flows this way. Given source v_0 and destination v_1, we want to use a Q with drift$(Q) = (v_2 - v_1)/N$. We get Q from the disintegration

$$Q^0 = \int Q \ \psi(dQ)$$

where ψ is a p.m. whose pushforward under the map

$$Q \to \text{drift}(Q) \bmod (1, 1)$$

is uniform on the continuous torus $[0, 1)^2$.

Putting together all source-destination pairs, we have flows on the skeleton graph which are independent of the realization of environment and for which the expectation of transportation measure across a $M \times M$ square equals Q^0.

Within each $M \times M$ square we simply use the flow $\mathbf{f}$ attaining

$$\text{inf}\{\text{cost}_M(\mathbf{f}, \mathbf{c}) : \ \mathbf{f} \text{ has t.m. } Q^0\}$$

Discussion

1. Much of the outline seems robust to model details. Plausible that method works quite generally to prove existence of limit constant for cost of optimal flows in $N \times N$ square with some kind of random environment. Essential requirement is

- Global cost function equals sum of order-N^2 local cost functions.

2. Assumed edge-capacity $= B$ (large); helps in some places, hinders in other places. Probably not hard to remove assumption.

3. In our open problem (maximum flow volume subject to i.i.d. edge-capacities) the C.I. fails. Because maximum flow volume across $M \times M$ square is order M but depends on order M^2 random variables.

4. The global optimal flow satisfies a certain condition by virtue of being a local minimum of

$$\mathbf{f} \to \mathsf{cost}_{(N)}(\mathbf{f}, \mathbf{c}).$$

But I don't know how to exploit this.