Almost everything so far has had two special features

- graph is a tree or complete graph

- explicit probability model on graph.

Lecture 4 describes briefly three quite different-looking topics without those features, each representative of some broader topic. All the proofs are very special to the particular topic, but the results illustrate the breadth of usefulness of LWC.

Topic 1: What is an infinite planar graph?

Recall the notion of a **finite planar graph**: can be embedded into the plane so that edges do not cross. One can use the same definition for **infinite** graphs, implying that the infinite 3-regular tree is planar. But intuitively this seems wrong – we expect our notion of "infinite planar graph" to be something like an inhomogeneous analog of some planar lattice.

Consider the question: is the rooted infinite 3-regular tree a local weak limit of finite G_n, for some choice of

- trees G_n? – no
- graphs G_n – yes
- planar graphs G_n - no.

The class of infinite (random) graphs **defined** as "local weak limits of finite (random) planar graphs" is therefore a subclass of "infinite planar graphs" but seems a natural class to study.

There seems to be exactly one known interesting result:

Theorem: (Benjamini and Schramm, 2001). Simple random walk on a (random) graph in this class with bounded degree is recurrent.

The proof is indirect, via disk-packing, and "non-quantitative" – I won't elaborate. But note that the "qualitative" conclusion of recurrence implies some more quantitative result, as follows.

Recall recurrence is equivalent to

$$E_i(\text{number of returns to } i) = \infty.$$

Write $r(\Delta, t)$ for the inf, over all finite planar graphs with degrees bounded by Δ , of

$$E_U(\text{number of returns to } U \text{ before time } t)$$

where U is a uniform random vertex. Then a compactness (of the space of rooted graphs with bounded degree) argument shows

Corollary $r(\Delta, t) \to \infty$ as $t \to \infty$.

Is there some more probabilistic argument that gives an explicit lower bound, say of order $t^{1/2}$?

[blackboard] Intuition: limit graphs can be "more recurrent" than lattice but not "less recurrent".

Open problems: what can you say about other "interacting particle systems" models on this class of graphs?

Topic 2: Counting quantities associated with a finite deterministic graph.

An **independent set** is a graph G is a subset of vertices, no two of which are adjacent. Write

$$I(G) = \text{ number of independent sets.}$$

Theorem: (Gamarnik - Bandyopadhyay, 2008). If G_n converges (LWC) to the infinite r-regular tree ($2 \leq r \leq 5$) then

$$n^{-1} \log I(G_n) \to \log c(r)$$

where $c(r) = x^{-r/2}(2-x)^{-(r-2)/2}$ and x solves $x^{-1} = 1 + x^{r-1}$.

Conceptual point: $I(G)$ is a "global" property of G but to first order (log scale) is determined by "local" behavior.

[blackboard:] in part similar to our CO arguments.

Another example within same topic

Theorem: (Lyons 2005). Write $\tau(G)$ = number of spanning trees of a finite G. If $G_n \to_{LWC} G$ where G is a deterministic infinite rooted graph, then

$$n^{-1} \log \tau(G_n) \to h(G)$$

where the limit **tree-entropy** $h(G)$ depends only on G.

Same conceptual point: $\tau(G)$ is a "global" property of G but to first order (log scale) is determined by "local" behavior.

Proof involves techniques from a large well-studied circle of ideas relating spanning trees, random walk, electrical networks.

One expression for the limit constant is

$$h(G) = \log(\text{degree}) - \sum_{k \geq 1} k^{-1} P(T = k)$$

where T is first return time for random walk on G.

Topic 3: uniform distributions on other combinatorial structures.

A **quadrangulation** is a planar graph in which each face has 4 edges. Visualize as [figure 1] though, to get a nice counting formula we allow multiple edges and leaves, as in [figure 2]. This allows the formula (Tutte, 1963): Number of rooted quadrangulations with n faces equals

$$\frac{2}{n+2}\,\frac{3^n}{n+1}\,\binom{2n}{n}.$$

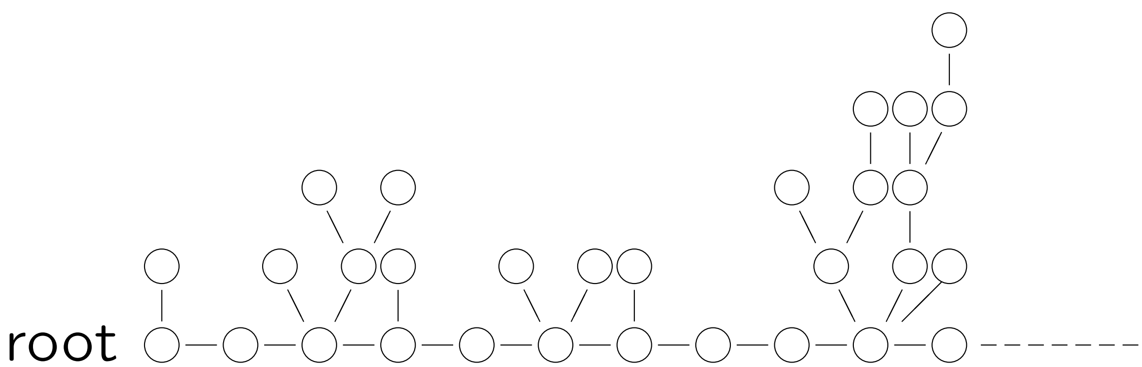
Seeing a simple formula like this suggests there is some bijective proof, enabling a construction of **uniform random quadrangulation** from simpler random objects. In this case it turns out there is a useful bijection from the set of n-vertex rooted quadrangulations to the following set.

Take a n-vertex rooted ordered tree ("birth order matters"). Assign label 0 to root and a label from $\{1, 2, 3, \ldots\}$ to other vertices, in such a way that the two labels at end vertices of an edge differ by exactly 1 ("tree-indexed simple walk").

The bijection takes one quadrangulation to one labeled tree and also gives a bijection between their vertices. The key feature of the bijection is that graph-distance from a vertex in the quadrangulation to the root, equals the label on the corresponding vertex of the labeled tree.

[example on blackboard]

Chassaing - Durhuus (2006) study local weak limits of the uniform random quadrangulation via local weak limits of the labeled tree. The structure of the limit tree is qualitatively the same as for unlabeled trees.



They show that, in the limit of of the uniform random quadrangulation, the number of vertices within distance r of the root grows as order r^4.