Recall from Lecture 1:

Convergence $G_n \to G_\infty$ of locally finite rooted graphs means:

for each fixed r, for $n > n_0(r)$ there is an isomorphism between Ball$(G_n; r)$ and Ball$(G_\infty; r)$.

Local weak convergence $G_n \to_{LWC} G_\infty$ means

$$G_n[U_n] \quad \stackrel{d}{\to} \quad G_\infty$$

where G_n is a random n-vertex unrooted graph and U_n a uniform random root.

By definition LWC gives first-order limits for any "local statistic" of the network, for instance

$$n^{-1}E(\text{number of degree-}j\text{ vertices of }G_n)$$
$$\to P(\text{root-degree}(G_\infty) = j)$$

$$n^{-1}E(\text{number of triangles in }G_n)$$
$$\to \tfrac{1}{3}E(\text{number of triangles in }G_\infty\text{ containing root})$$

under a UI condition.

Lecture 2: An "Elementary" Illustration of the Local Weak Convergence Methodology to Combinatorial Optimization over Random Data

The topic is **Maximum Partial Matchings on Random Trees:** from Aldous-Steele (2003). Start with an artificial deterministic CO problem.

- n-vertex graph G
- positive weights ξ_e on edges e
- partial matching S is a vertex-disjoint set of edges, and its weight is $\text{wt}(S) = \sum_{e \in S} \xi_e$.

So associated with the weighted graph G is a number

$$M(G) = \max\{\text{wt}(S) : \ S \text{ a partial matching } \}.$$

We study this under the **Probability Model**:

- Edge-weights (ξ_e) are i.i.d.
  $\xi > 0$, $E\xi < \infty$, dist(ξ) non-atomic.

- G is a random tree T_n uniform on all n^{n-2} trees on vertex-set $[n]$.

It takes 10 seconds to conjecture
Theorem $n^{-1}EM(T_n) \to c$ as $n \to \infty$
where the limit constant c depends on dist(ξ).

We will outline a proof and a fairly explicit characterization of c.

Note **intuition:** The deterministic CO problem on a tree should be easy – recurse somehow – and similarly the random setting should have some recursive decomposition (cf. problems on Galton-Watson BPs). So what exactly is this recursion?

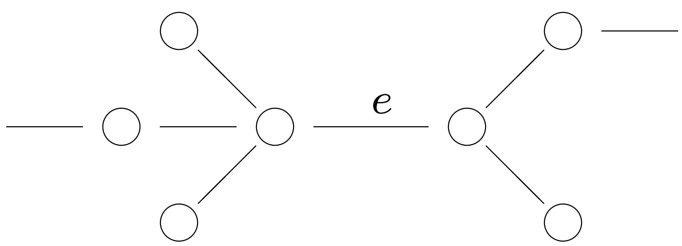
(Deterministic setting:)
On a rooted weighted tree T define the bonus $B(T)$ by

$$\begin{aligned} B(T) &= \text{max weight of matching} \\ &- \text{max weight of matching not using root.} \end{aligned}$$

An arbitrary edge e of T defines two rooted subtrees – call them $T^{big}(e)$ and $T^{small}(e)$. An elementary argument [on blackboard] gives an explicit criterion for whether e is in the max-weight matching $\mathcal{M}$:

$$e \in \mathcal{M} \text{ iff } \xi_e > B(T^{small}(e)) + B(T^{big}(e))$$

(assuming cannot have =).



The maximum weight $M(T)$ is obtained by summing ξ_e over $e \in \mathcal{M}$. Rephasing in terms of a uniform random edge $\mathbf{e}$ leads to

$$\frac{M(T)}{n-1} = E\left[\xi_{\mathbf{e}}\ 1(\xi_{\mathbf{e}} > B(T^{small}(\mathbf{e})) + B(T^{big}(\mathbf{e})))\ \right]$$

where the only randomness is from "uniform random edge $\mathbf{e}$".

Now take some probability model for a random n-vertex tree T_n, and i.i.d. edge-weights (ξ_e). Then $\xi_{\mathbf{e}}$ is independent of $\left(B(T_n^{small}(\mathbf{e})),\ B(T_n^{big}(\mathbf{e}))\right)$. So if we can prove joint convergence

$$\left(B(T_n^{small}(\mathbf{e})),\ B(T_n^{big}(\mathbf{e}))\right) \ \stackrel{d}{\to}\ (Y, Z) \quad (*)$$

and identify the limit distribution, then we have proved the Theorem:

$$\lim_n n^{-1} EM(T_n) = E[\xi\ 1(\xi > Y + Z)]$$

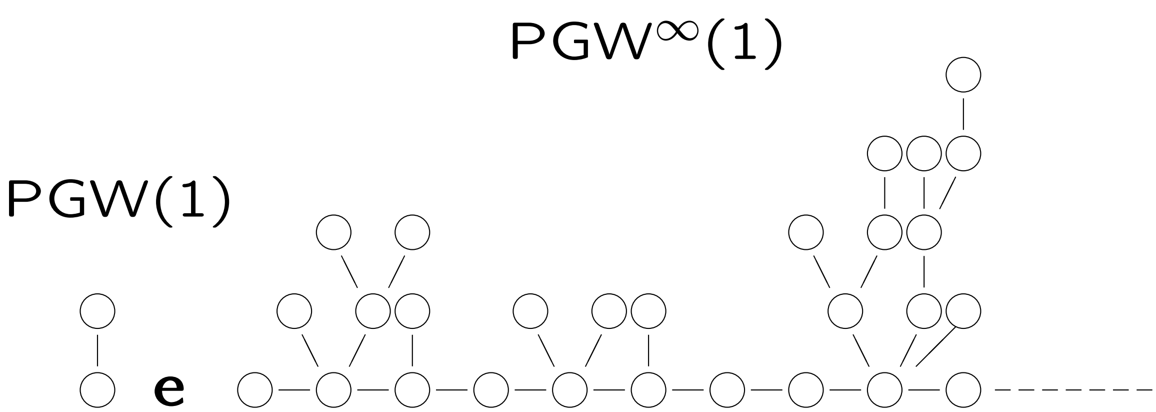
with ξ independent of (Y, Z).

Note we have not yet appealed to a particular model of random n-vertex trees T_n.

But are we getting anywhere? We have just reduced study of expectation of $M(T_n)$ to study of distribution of $B(T_n^{big}(\mathbf{e}))$ etc which is a (small) difference of two (large) random quantities. These look like harder questions!

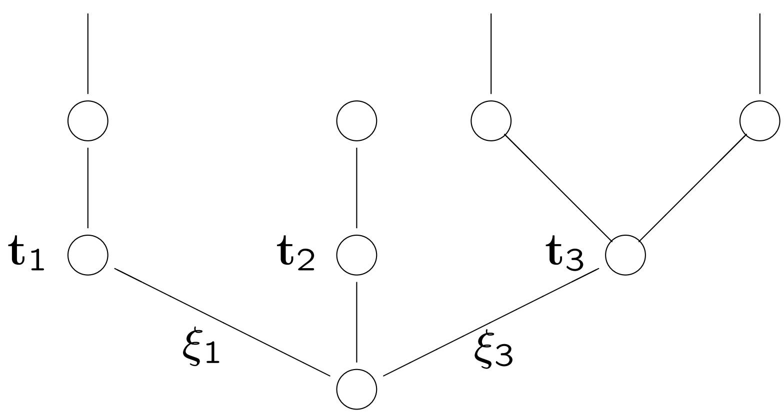
Let us forget "bonus" for a minute; think about the subtrees themselves. For the particular model "a random tree T_n uniform on all n^{n-2} trees on vertex-set $[n]$" we saw in Lecture 1 that there is the following $n \to \infty$ LWC limit :

$$\begin{aligned}(T_n^{small}(\mathbf{e}), T_n^{big}(\mathbf{e})) &\xrightarrow{d} (T_\infty^{small}, T_\infty^{big}) \\ &\stackrel{d}{=} \text{PGW}(1) \times \text{PGW}^\infty(1).\end{aligned}$$



Look first at small subtrees. Certainly we have $B(T_n^{small}) \xrightarrow{d} B(T_\infty^{small})$. Although the realizations of T_∞^{small} can be any finite tree $\mathbf{t}$, we don't need to calculate $B(\mathbf{t})$ for every finite tree, because we have a recursion. For a finite tree $\mathbf{t}$ with first generation subtrees $\mathbf{t}_i$,

$$B(\mathbf{t}) = \max(0; \xi_i - B(\mathbf{t}_i), i \geq 1).$$



Apply to T_∞^{small}: bonus $Y = B(T_\infty^{small})$ satisfies

$$Y \stackrel{d}{=} \max(0, \xi_i - Y_i,\ 1 \leq i \leq \text{Pois}(1))$$

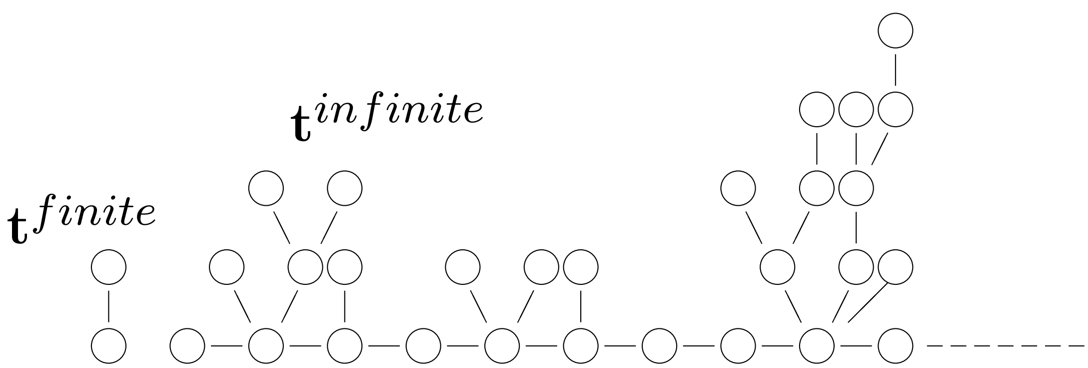
where $(\xi_i, i \geq 1)$ are i.i.d. with given weight distribution, and $(Y_i, i \geq 1)$ are i.i.d. with the unknown distribution of Y. This is a prototype of a **recursive distributional equation**. In this context, knowing tree is a.s. finite, easy to show $\exists$ unique solution Y, depending on dist(ξ).

Recall $T_n^{big} \xrightarrow{d} T_\infty^{big}$, where limit T_∞^{big} is a infinite sequence of PGW(1) trees attached to a baseline. What about the bonuses?

$$B(T_\infty^{big}) = \infty - \infty = \quad ?????$$

But suppose heuristically $B(\mathbf{t})$ makes sense for suitable infinite trees. Again we get a recursion, which now is

$$B(\mathbf{t}) = \max(\xi - B(\mathbf{t}^{infinite}), B(\mathbf{t}^{finite})).$$



Apply to T_∞^{big}: suggests $Z = B(T_\infty^{big})$ satisfies

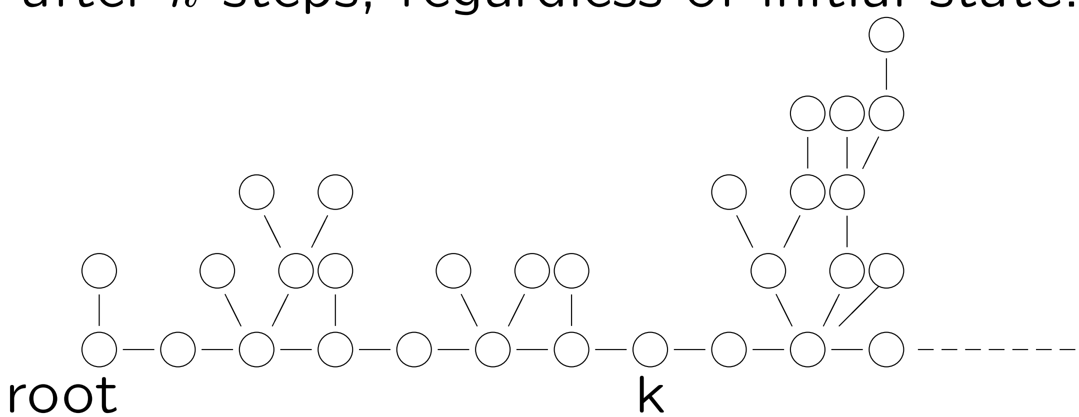
$$Z \stackrel{d}{=} \max(\xi - Z', B(T_\infty^{small}))$$

where $B(T_\infty^{small}) \stackrel{d}{=} Y$ and ξ independent, known distributions; $Z' \stackrel{d}{=} Z$, unknown distribution.

1-slide outline of rigorous analysis here (not representative of general methodology).
We recognize equation

$$Z \overset{d}{=} \max(\xi - Z', Y)$$

as equation for stationary distribution of the Markov chain $z \to \max(\xi - z, Y)$ on $[0, \infty)$. For this special chain, easy to give quantitative bounds: within $\delta(k)$ of stationary distribution after k steps, regardless of initial state.



For large n, one step of the chain represents (approx) how $B(T^{big})$ changes as a root is moved one step to the left along baseline. Fix k. We can't directly analyze the $n \to \infty$ limit of $B(T^{big})$ at baseline position k. But we don't need to; the mixing of the chain ensures we are within $\delta(k-1)$ of stationary distribution at position 1.

So we have proved

Theorem $n^{-1}EM(T_n) \to c = E[\xi \, 1(\xi > Y+Z)]$ where Y and Z are solutions of

$$Y \stackrel{d}{=} \max(0, \xi_i - Y_i, \ 1 \le i \le \text{Pois}(1))$$

$$Z \stackrel{d}{=} \max(\xi - Z', Y).$$

Can do explicit calculations in the special case where ξ has exponential(1) distribution:

$$\begin{aligned} P(Y \le y) &= \exp(-ce^{-y}), \quad y \ge 0 \\ P(Z \le z) &= (1 - be^{-z}) \ \exp(-ce^{-z}), \quad z \ge 0 \end{aligned}$$

where $c \approx 0.7146$ is the strictly positive solution of $c^2 + e^{-c} = 1$ and $b = \frac{c^2}{c^2+2c-1} \approx 0.5433$. So

$$\lim_n n^{-1}EM(T_n) =$$

$$\int_0^\infty se^{-s} ds \int_0^s c(e^{-y} - be^{-s}) \exp(-ce^{-y} - ce^{-(s-y)}) \, dy$$

$$\approx 0.2396.$$

This example illustrates the two steps common to uses of LWC in CO problems over random data.

An inclusion criterion for whether an edge is in the optimal solution. This was explicit-implicit; an explicit condition in terms of implicitly-defined other quantities.

Recursive distributional equations. When the limit structure is tree-like, these other quantities of interest can often be expressed in terms of solutions of RDEs. That is, an "unknown" distribution X satisfies

$$X \stackrel{d}{=} g(\xi_i, X_i, 1 \le i \le N)$$

where the distribution $(N; \xi_i, 1 \le i \le N)$ and the function $g(\cdot)$ are given, and $(X;\ X_i, i \ge 1)$ are i.i.d.

xxx show table AB.pdf

Lecture 2, part 2. Move on to a slightly more sophisticated setting: the "mean-field model of distance" – will continue into Lecture 3.

Recall **Yule process**. One individual at time 0. Each individual v gives birth at ages $\xi_{v1}, \xi_{v2}, \ldots$ distributed as a Poisson (rate 1) process on $(0, \infty)$.

Associated tree typically drawn (blackboard) on "absolute time" scale. We want to view it as a "spatial" tree; when individual v gives birth to w at age ℓ, draw an edge (v, w) of **length** ℓ. Call this the PWIT (Poisson weighted infinite tree). It's best viewed as a Java simulation.

Take the complete graph on n vertices. To each of the $\binom{n}{2}$ edges e assign i.i.d. lengths ℓ_e with Exponential(mean n) distribution. Call this network G_n. Easy to see:

$$G_n \to_{LWC} PWIT.$$

This provides one motivation for study of PWIT. More generally, for problems about random points in d dimensions where one wants to calculate numbers – which is usually impossible – the PWIT provides an artificial but mathematically tractable alternative – a "mean-field model of distance".

Illustrate by outlining proof of a celebrated result of Frieze. Writing MST for **minimum spanning tree**,

$$n^{-1} E \text{ len}(MST(G_n)) \to \zeta(3).$$

Consider a finite network, generic edge-lengths. The MST is by definition the ST of minimum total length; elementary that it can be found by either a local or a global greedy algorithm. Here's a slight variant of the textbook lemma.

Lemma (inclusion criterion). Define $\mathfrak{t}$ as the set of edges (v, w) with the property:
there does **not** exist a path $v = v_0, v_1, v_2, \ldots, v_k = w$ such that each $\ell(v_i, v_{i+1}) < \ell(v, w)$.
Then $\mathfrak{t}$ is the MST.

[blackboard – this **does** define a ST.]

Useful to restate Lemma as follows. Network G, vertex v, length s. In the subnetwork of edges with length $< s$, write $c(G, v, s)$ for the component containing v.

Lemma (inclusion criterion). In a finite network, the MST is the set of edges (v, w) with the property:

$$c(G, v, \ell(v, w)) \neq c(G, w, \ell(v, w)). \qquad (1)$$

On a typical infinite network, all STs will have infinite length, so we don't have an obvious notion of minimum-length ST. But we can use criterion (1) to define an object called the minimum spanning **forest**. We consider only the PWIT.

Definition. MSF(PWIT) is the set of edges (v_1, v_2) such that
(i) $c(PWIT, v_1, \ell(v_1, v_2)) \neq c(PWIT, v_2, \ell(v_1, v_2))$
(ii) these two components are not both infinite.

Heuristically, this is the "wired at infinity" ST. From definition, easy to check it is a spanning forest whose tree-components are all infinite. From Java simulation, we see it's not a single tree (paths directed to infinite boundary).

We observed before that, for $G_n =$ complete graph with random edge-lengths,

$$G_n \to_{LWC} PWIT.$$

Now we assert more:

$$(G_n, MST(G_n)) \to_{LWC} (PWIT, MSF(PWIT))$$

where e.g. we indicate edges of MST by $0-1$ marks.

How to prove

$(G_n, MST(G_n)) \to_{LWC} (PWIT, MSF(PWIT))$?

If an edge of the PWIT is in the MSF, this fact is "witnessed" by some finite region, and so the corresponding edge must eventually be in $MST(G_n)$. Converse not obvious; but enough to show that mean degree of MSF equals 2, which is consequence of explicit formulas below.

To exploit this convergence, start (as with the "partial matching" example) by expressing $E\text{len}(MST(G_n))$ in terms of a uniform random vertex U_n. For any finite network,

$$2 \text{ len}(MST)) = \sum_v \sum_e \ell(e) 1_{(e \in MST)} 1_{(v \in e)}$$

and in terms of the uniform random vertex U_n

$$\frac{2}{n} \text{ len}(MST)) = E \sum_e \ell(e) 1_{(e \in MST)} 1_{(U_n \in e)}$$

$$= E \sum_{e \ni U_n} \ell(e) 1_{(e \in MST)}.$$

The right side is "local" around U_n, so applying to G_n, the LWC gives

$$2 \lim_n \frac{1}{n} E \text{ len}(MST(G_n)) = E \sum_{e \ni \text{root}} \ell(e) 1_{(e \in MSF)}$$

up to checking UI. The right side refers to the PWIT, and so Frieze's theorem reduces to explicit calculations with the PWIT.

Now the edge-lengths at the root form a rate-1 Poisson process, so by conditioning on length s we can write

$$E \sum_{e \ni \text{root}} \ell(e) 1_{(e \in MSF)} = \int_0^\infty h(s) \ s \ 1 \ ds$$

where $h(s)$ is the chance that a length-s edge is in the MSF. By a familiar property of the Poisson process, after conditioning on existence of an edge of length s, the other points are Poisson distributed, so conditionally we see two independent copies of the PWIT linked by an edge of length s. [blackboard]. The subtree of a PWIT consisting of edges of length $< s$ is just $PGW(s)$, so by definition of MSF

$$h(s) = 1 - q^2(s) \text{ where}$$

$$q(s) = P(PGW(s) \text{ is infinite}).$$

Now reduced to a hard homework problem. We've got the limit constant c as

$$c = \tfrac{1}{2} \int_0^\infty s(1 - q^2(s)) \ ds$$

and we know from BP theory that $q(s)$ = chance of non-extinction satisfies a certain equation, which works out to be

$$q(s) = 1 - \exp(-sq(s)).$$

Integrate by parts

$$c = \tfrac{1}{2} \int_0^\infty s^2 q(s) q'(s) \ ds$$

write s in terms of q

$$s = \tfrac{-\log(1-q)}{q}$$

and change variables to q; we find

$$c = \tfrac{1}{2} \int_0^1 \tfrac{q \log^2(1-q)}{q^2} \ dq.$$

Change variables again to $u = -\log(1 - q)$:

$$\begin{aligned} c &= \tfrac{1}{2} \int_0^\infty u^2 \tfrac{e^{-u}}{1-e^{-u}} \ du \\ &= \tfrac{1}{2} \int_0^\infty \sum_{i \geq 1} u^2 e^{-iu} \ du \\ &= \tfrac{1}{2} \sum_{i \geq 1} 2 i^{-3}. \end{aligned}$$

The technical gap is a UI argument to go from

$$\sum_{e \ni U_n} \ell(e) 1_{(e \in MST(G_n))} \stackrel{d}{\to} \sum_{e \ni \text{root}} \ell(e) 1_{(e \in MSF(PWIT))}$$

to convergence of expectations.

If UI automatically held, then for any sequence G_n^* converging LWC to the PWIT, the mean length of $MST(G_n^*)$ would grow as $\zeta(3)n$.

But this isn't true, e.g [blackboard] for two complete graphs connected by a single edge. So do need a technical argument (in this case, already known as part of Erdos-Renyi component size analysis).