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Consider the Smoluchowski coagulation equation, with $n(x,0) = 1_{(x=1)}$. Say that $K(x,y)$ exhibits **gelation** (a phase transition) if the average mass density

$$\sum_{x=1}^{\infty} xn(x,t) \begin{array}{ll} = 1 & , \ t < T_{\text{gel}} \\ < 1 & , \ t > T_{\text{gel}} \end{array}$$

where $T_{\text{gel}} < \infty$. In words, "some proportion of mass enters infinite clusters" after time T_{gel}. Curiously, this phase transition is not well understood at a rigorous level.

Suppose kernel has scaling exponent γ:

$$K(cx, cy) = c^{\gamma} K(x,y).$$

(rigorous)	$\gamma \leq 1 \Rightarrow$	no gelation
(heuristic *)	$\gamma > 1 \Rightarrow$	gelation

(*) recent partial results in Jeon (1998).

Note the "formation of planets" example had $\gamma = 4/3$.

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Open problems for $N \to \infty$ behavior of the Marcus-Lushnikov process.

$$K(cx, cy) = c^{\gamma} K(x, y).$$

Non-gelling case $\gamma \le 1$.

(Time until largest cluster size $\ge N/2$)

$$\begin{aligned} &\approx N^{1-\gamma} \qquad (\gamma < 1) \\ &\approx \log N \qquad (\gamma = 1) \end{aligned}$$

At this time, second-largest cluster size is $\Omega(N)$.

Gelling case $1 < \gamma \le 2$.

$$\begin{aligned} \text{(largest cluster size at } t) &= o(N^{\varepsilon}),\ t < T_{\text{gel}} \\ &= \Omega(N),\ t > T_{\text{gel}} \end{aligned}$$

(size of emerging giant cluster *) $\approx N^{\frac{2}{1+\gamma}}$

* formalize by e.g.
$\sup_t$ (size of second-largest cluster at time t).

Speculation about general gelling kernels is based on the well-understood behavior of one special case, the multiplicative kernel $K(x,y) = xy$. Here we shall see that the Marcus-Lushnikov process becomes the classical Erdös - Rényi random graph process; more precisely, the size of connected components in the random graph process.

Recall the **Random graph** $\mathcal{G}(n, p(n))$ has n vertices, and each potential edge is present with probability $p(n)$.

Take $p(n) = \lambda/n$. Then each vertex has mean $\sim \lambda$ edges, and the graph looks locally like a Galton-Watson branching process with Poisson(λ) offspring distribution.

By the elementary subcritical/supercritical properties of the GWBP, we expect different behavior for $\lambda < 1$ than for $\lambda > 1$. Take $p(n) = \lambda/n$. Write $C_1(n) \geq C_2(n) \geq \ldots\ldots$ for the component sizes. Well known results in random graph theory:

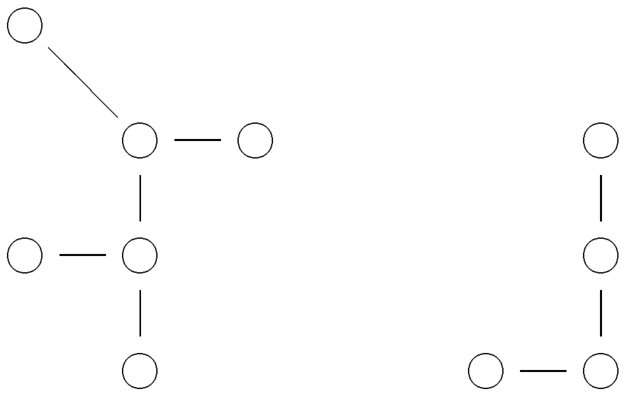
($\lambda < 1$): $C_1(n)$ and $C_2(n) \sim c(\lambda) \log n$

($\lambda > 1$): $C_1(n) \sim c(\lambda)n, \ C_2(n) \sim c'(\lambda) \log n$

($\lambda = 1$) : $C_1(n)$ and $C_2(n) = \Theta(n^{2/3})$, random.

To probe more deeply into behavior near the "critical point" $p(n) = 1/n$, fix $-\infty < t < \infty$ and consider $p(n) = \frac{1}{n} + \frac{t}{n^{4/3}}$. The exponent $n^{4/3}$ here relates to the exponent $n^{2/3}$ in the size, as follows.

$$\mathcal{G}(n, \frac{1}{n} + \frac{t}{n^{4/3}})$$



size $xn^{2/3}$ size $yn^{2/3}$

When we increment t to $t + dt$, each of the $xyn^{4/3}$ possible edges appears with probability $dt/n^{4/3}$. and so there is rate $xy\ dt$ of merging into a component of size $(x + y)n^{2/3}$.

This suggests the following.

Theorem. Write $C_1(n,t) \geq C_2(n,t) \geq \ldots$ for component sizes of $\mathcal{G}(n, \frac{1}{n} + \frac{t}{n^{4/3}})$. Then

$$n^{-2/3}(C_1(n,t), C_2(n,t), \ldots) \ \stackrel{d}{\to}$$

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots)$$

where the limit process, indexed by $-\infty < t < \infty$, evolves as the **multiplicative coalescent**:

For each pair of clusters, masses $\{x_i, x_j\}$ say, they coalesce into one cluster of mass $x_i + x_j$ at rate $x_i x_j$.

Proved in Aldous (1997) – outlined next. Note strange feature: the random graph has n vertices, but we rescale size by only $n^{2/3}$, so total mass $\to \infty$. For the multiplicative coalescent the state space is

$$\{\mathbf{x} = (x_1, x_2, \ldots) : \sum_i x_i^2 < \infty\}.$$

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Inhomogeneous reflecting Brownian motion $(B^t(s), 0 \leq s < \infty)$ with drift $= t - s$ at time s.

$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots)$ is the ranked list of lengths of excursions from 0 of $B^t(\cdot)$.

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As time t increases from $-\infty$ to $+\infty$ the size of the largest cluster grows from 0 to ∞.

Step i of walk:

$X(i) - X(i-1) = -1 +$ # children of vertex i.

Size of j'th component $= T_j - T_{j-1}$

$$T_j := \min\{i : X(i) = -j\}.$$

If original graph is from the probability model $\mathcal{G}(n, \frac{1}{n} + \frac{t}{n^{4/3}})$ then we can write exact expressions for conditional mean and variance step size.

$$\begin{aligned}
\text{cond. mean} &= -1 + p(n) \times (\#\text{ available vertices}) \\
&\approx -1 + \left(\frac{1}{n} + \frac{t}{n^{4/3}}\right)(n - i) \\
&\approx \frac{t - in^{-2/3}}{n^{1/3}}. \\
\text{cond. var.} &\approx 1.
\end{aligned}$$

Rescale:

$$\text{(“time”)} \quad i \leftarrow sn^{2/3}$$
$$\text{(size)} \quad x \leftarrow yn^{1/3}$$

For rescaled process, cond. mean drift rate

$$= \frac{t-s}{n^{1/3}} \times n^{2/3} \times n^{-1/3} = t - s$$

time change   size change

cond. variance rate

$$= 1 \times n^{2/3} \times (n^{-1/3})^2 = 1$$

time change   size change

Then standard results on martingales $\rightarrow_{dist}$ Brownian motion imply limit process is the inhomogeneous Brownian motion $(B^t(s), 0 \leq s < \infty)$ with drift $= t - s$ at time s.

Open problems for general stochastic coalescents

1. Give general conditions on $K(x, y)$ which imply (e.g.) the Feller property.

2. Study the entrance boundary at time $-\infty$. In other words, describe all the ways that the process can start from infinitesimally small clusters at time $-\infty$.

Comments. For the additive and the multiplicative kernels there are ad hoc proofs of the Feller property. For these kernels the entrance boundary is surprisingly rich [Aldous - Limic (1998), Aldous - Pitman (1999)] but we think this is anomalous, and that for typical kernels there is a unique way to come in from time $-\infty$.

Another type of question is to justify these mean-field models as limits within 3-dimensional statistical physics models in which positions and velocities/diffusion rates are completely specified. Surprisingly, this hasn't been done even for Smoluchowski's original Brownian motion setting, although Lang - Nguyen (1980) treat the mathematically simpler case where diffusion rate doesn't depend on size.