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A Survey of Max-Type Recursive Distributional Equations

Based on paper in preparation.

Prototype Example. Subcritical Galton-Watson BP with N offspring. Suppose interested in

$$Z := \text{ total population } .$$

Then

$$Z = 1 + \sum_{i=1}^{N} Z_i$$

where Z_i is subpopulation derived from child i. So Z satisfies the recursive distributional equation (RDE)

$$Z \ \stackrel{d}{=} \ 1 + \sum_{i=1}^{N} Z_i \quad (Z_i \text{ i.i.d. copies of } Z).$$

We write the general RDE as

$$X \stackrel{d}{=} g(\xi; X_i, i \geq 1)$$

More elaborate example – Rösler (1992) in context of Quicksort

$$X \stackrel{d}{=} UX_1 + (1-U)X_2 + C(U); \quad U \stackrel{d}{=} \text{uniform}(0,1)$$

$$C(x) := 2x \log x + 2x \log(1-x) + 1.$$

Can regard as fixed point equation for an unknown distribution dist(X) on $\mathbb{R}$. Elementary contraction argument proves unique solution. The "positive linear RDE"

$$X \stackrel{d}{=} \sum_i A_i X_i; \quad A_i, X_i \geq 0$$

well understood – Durrett and Liggett (1983). In this talk:

- a theoretical perspective

- 3 examples of "max-type" g.

Abstract setup. Take $\mathcal{P}$ as set of prob. dists. on a state space S. Take a distribution dist(ξ) on some other space. Take S-valued function $g(\cdot)$ with appropriate domain. Consider expression

$$\widehat{X} = g(\xi; X_i, i \geq 1); \quad X_i \text{ i.i.d. } (\mu). \tag{1}$$

Can regard expression as a map $\mathcal{P} \to \mathcal{P}$

$$\mu \to T(\mu) := \text{dist}(\widehat{X}).$$

Thus one can interpret

$$X \ \stackrel{d}{=} \ g(\xi; X_i, i \geq 1)$$

as the fixed point equation $\mu = T(\mu)$. But we can also interpret (1) in terms of r.v.'s as a typical step in a <u>recursive tree process</u> (RTP).

xxx Figure 1

invariant RTP (recursive tree process)

$\Updownarrow$

solution of RDE (recursive distributional equation)

Statistical physics viewpoint. In a RTP, the value $X_\emptyset$ at the "root" $\emptyset$ is determined by (for each h)

- the values $\xi_\mathbf{i}$ of the "innovations" at points $\mathbf{i}$ in the first h generations

- the "boundary values" $X_\mathbf{i}$ for $\mathbf{i}$ in generation h.

Definition 1 *An invariant RTP is* <u>*endogenous*</u> *if $X_\emptyset$ is measurable w.r.t. ($\xi_\mathbf{i}$: all descendants $\mathbf{i}$).*

Easy examples show that one cannot determine whether this property holds just by looking at the induced map T on probability distributions.

Example 1. Consider branching random walk (BRW) on $\mathbb{R}^1$; parametrized by $\mathrm{dist}(\xi_i, i \geq 1)$, positions of offspring relative to parent.

Study R_n = position of rightmost individual in generation n. Biggins (1977): under mild conditions, there is a constant γ, computable from $\mathrm{dist}(\xi_i, i \geq 1)$,

$$n^{-1} R_n \to \gamma.$$

Known, under too-restrictive conditions (e.g. (ξ_i) independent, log-concave density: Bachmann (2000))

$$\begin{aligned} \text{median}(R_{n+1}) - \text{median}(R_n) &\to \gamma \\ R_n - \text{median}(R_n) &\stackrel{d}{\to} X \end{aligned}$$

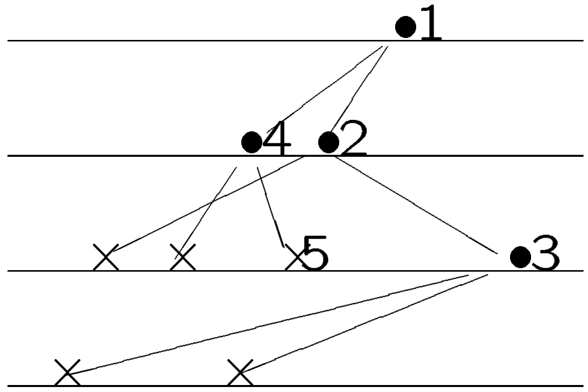
in which case X must satisfy the RDE

$$X \stackrel{d}{=} -\gamma + \max_i(\xi_i + X_i), -\infty < X < \infty. \quad (2)$$

Here the corresponding invariant RTP is <u>not</u> endogenous.

It would be interesting to study directly the issue of technical conditions needed to ensure existence/uniqueness of solution of (2), separate from the weak convergence issue.

Example 2. Same setting, with $\gamma > 0$. Now think of BRW tree as "data"; consider algorithmic question of finding individuals at large positive positions. Inefficient to look at all individuals. Natural greedy algorithm studied in Aldous (1992).



Study position X_n of n'th individual examined by greedy algorithm. Expect a limit

$$n^{-1} X_n \to \text{speed}.$$

Turns out (not obvious) in independent binary case

$$\text{speed} = E(\xi + L)^+$$

for $-\infty < L \leq 0$ defined by

$$L = \sup_{(w_i)} \inf_i \text{ (position of } w_i) \qquad (3)$$

the *sup* over lines of descent (w_i). This L satisfies the RDE

$$L \stackrel{d}{=} \min(0, \max_i (L_i + \xi_i)), \qquad -\infty < L \leq 0.$$

Here one can show L is the unique solution of the RDE; moreover by the a.s. formula (3) the invariant RTP must be endogenous.

Example 3. Frozen percolation on the binary tree $\mathbb{T}_3$. Attach independent uniform$(0,1)$ times U_e to edges e. Initially each edge e is OFF. Edge e turns ON at time U_e <u>unless</u> it is adjacent to some infinite ON-component.

In other words, an ON-component grows to infinite size but then "freezes" and doesn't grow further. The ultimate ON-set $\mathcal{C}_\infty$ contains both finite and infinite components.

Aldous (2000): process exists on $\mathbb{T}_3$, can do explicit calculations e.g.
$P(e$ in finite component of $\mathcal{C}_\infty) = 1/16$
$P(e$ in infinite component of $\mathcal{C}_\infty) = 7/12$.

Benjamini–Schramm (p.c.) show process does not exist on $\mathbb{Z}^2$.

Consider modified tree with a degree-1 root. Let X be time at which root joins infinite component ($= \infty$ if never). Easy to see a recursion in terms of X_1, X_2 defined similarly for subtrees:

$$X = \Phi(\min(X_1, X_2), U_e)$$

$$\begin{aligned} \Phi(x, u) &= x \text{ if } x > u \\ &= \infty \text{ if } x < u. \end{aligned}$$

This RDE has solution

$$f_X(x) = \frac{1}{2x^2}, \tfrac{1}{2} < x < 1; \;\; P(X = \infty) = \tfrac{1}{2}.$$

One can use the associated RTP to construct rigorously a version of the frozen percolation process on $\mathbb{T}_3$.

The property

(*) the RTP is endoegeneous

is equivalent to the property

(**) the frozen percolation process can be constructed from the given (U_e) without external randomization.

Numerics suggests this property is true.