The Asymmetric One-Dimensional Constrained Ising Model: Rigorous Results

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Abstract

We study a one-dimensional spin (interacting particle) system, with product Bernoulli(p) stationary distribution, in which a site can flip only when its left neighbor is in state $+1$. Such models have been studied in physics as simple exemplars of systems exhibiting slow relaxation. In our "East" model the natural conjecture is that the relaxation time $\tau(p)$, that is 1/(spectral gap), satisfies $\log \tau(p) \sim \frac{\log^2(1/p)}{\log 2}$ as $p \downarrow 0$. We prove this up to a factor of 2. The upper bound uses the Poincaré comparison argument applied to a "wave" (long-range) comparison process, which we analyze by probabilistic techniques. Such comparison arguments go back to Holley (1984, 1985). The lower bound, which atypically is not easy, involves construction and analysis of a certain "coalescing random jumps" process.

KEY WORDS. constrained Ising model, coupling, exponential martingale, Poincaré inequality, relaxation time, spectral gap.

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1 Introduction

The asymmetric one-dimensional constrained Ising model, or more briefly the *East process*, is an interacting particle system with sites $\mathbb{Z}^1$ and each site having two states $\{0,1\} = \{\text{unoccupied}, \text{occupied}\}$. Its essential qualitative feature is that a site can change state only when the site to its left is occupied. The flip rates at each site i are specified by:

• if site $i-1$ is in state 0 then state i cannot change;

• if site $i-1$ is in state 1 then state i flips $\begin{array}{ll} 0 \to 1; & \text{rate } p \\ 1 \to 0; & \text{rate } 1-p \end{array}$

where $0 < p < 1$ is a parameter. Here is an equivalent description. Each particle, at rate 1 (that is, the times of a Poisson(1) process) sends a "pulse" to the site to its East, and the state of that site is reset via

$$P(\text{occupied}) = p, \quad P(\text{unoccupied}) = 1-p.$$

A careful construction of the process is outlined in the Appendix. By routine arguments the i.i.d. Bernoulli(p) measure is the unique non-trivial stationary distribution and this *East process* is reversible. Write $(\mathbf{X}(t), 0 \le t < \infty) = ((X_i(t), i \in \mathbb{Z}), 0 \le t < \infty)$ for the stationary process. We are interested in studying the *relaxation time* (defined as 1/spectral gap), say $\tau(p)$, as $p \downarrow 0$. This specific process, and questions concerning its relaxation time, were introduced by Jäckle and Eisinger [10] and further studied in the physics literature in [6, 13, 14]. Sollich and Evans [18] argue non-rigorously that

$$\log \tau(p) \sim \frac{\log^2(1/p)}{\log 2} \text{ as } p \downarrow 0 \tag{1}$$

(*log* is natural logarithm), and observe that (1) is consistent with Monte Carlo simulations. Section 2.3 outlines a simple heuristic argument. The purpose of this paper is prove rigorous bounds which support this conjecture.

Theorem 1 *(a)* $\log \tau(p) \le \left(\frac{1}{\log 2} + o(1)\right) \log^2(1/p)$ *as* $p \downarrow 0$.
(b) $\log \tau(p) \ge \left(\frac{1}{2\log 2} - o(1)\right) \log^2(1/p)$ *as* $p \downarrow 0$.

The proof of the lower bound (b), outlined in section 5.1 with details in section 5.2, involves the usual method of applying the variational characterization (3) to a suitable test function g. But unusually, finding a good g is not intuitively simple, and in fact we define g only implicitly in terms of a certain *coalescing random jumps* process which we need to invent. Section 5.3 motivates this particular process. A more inspired choice of g might allow one to remove the factor $\frac{1}{2}$ from the lower bound and thus prove the conjecture (1).

The proof of the upper bound (a) uses the Poincaré comparison method, used by Holley [9, 8] and developed by Diaconis and Saloff-Coste [4], which bounds the relaxation time of one process in terms of the relaxation time of another "comparison" process with the same stationary distribution. Though the simplest comparison process would be the process in which sites flip independently, making the comparison with this process seems technically difficult. Instead, following an idea used by Holley [8], we use for comparison a certain "long range" process, which for us is (section 3) the following *wave* process, For each particle (at site i, say) at rate 1 a wave appears which, in the $10/p$ sites to the right of i, deletes existing particles and replaces them by an independent Bernoulli(p) family of particles. Proposition 7 shows that for small p the relaxation time of the wave process is bounded by the constant $10/3$. The proof (section 3.2) uses probabilistic methods: coupling and supermartingale estimates for the position of the rightmost particle in the wave process started from a single particle. Section 4 gives the argument comparing the East process with the wave process.

1.1 Remarks on related models and techniques

1. The East model is a special case of more general *constrained* or *facilitated* spin models on lattices, in which a site is permitted to flip only when some prescribed number of neighbors are in state 1. Such models go back to Frederickson and Anderson [7], and a recent review is in Pitts et al [15]. The models are intended to illustrate the slow relaxation behavior of liquids near the glass transition. Our methods are tied closely to the specifics of the East model, but making rigorous the treatments in [15] concerning other models would be an interesting challenge. A specific next model one might study is to take as site space the infinite rooted binary tree, and allow flips at a site only when both children of that site are in state 1. In this model it is easy to show

$$\tau(p) \geq c(p - \tfrac{1}{2})^{-2}, \ p > 1/2$$

and one would like to prove the corresponding upper bound. Apparently much harder is the *North-East model*, where site space is $\mathbb{Z}^2$ and flips are allowed at (x, y) only if sites $(x-1, y)$ and $(x, y-1)$ are both in state 1. Here one would like to find a critical exponent $\alpha > 0$ such that

$$\tau(p) \sim c(p - p_{\text{crit}})^{-\alpha} \text{ as } p \downarrow p_{\text{crit}}$$

where p_{crit} is the critical value for oriented percolation of 0-sites.

In the terminology of [15] our model should be called the *West* model; we reversed direction to make site labeling more natural.

2. Since Diaconis and Stroock [5] there have been many papers studying relaxation times for Markov chains on various "combinatorial" state spaces via the basic comparison method (implicitly comparing the given chain with the i.i.d. chain). Comparisons with some judiciously-chosen alternate chain were used in [4] to study the exclusion process, and in [16] for other statistical physics models. Martinelli [12] uses such arguments in studying the subcritical two-dimensional Ising process.

3. One might suppose that general results for one-dimensional Ising-type models implied

$$\tau(p) < \infty \text{ for each } p > 0 \tag{2}$$

But the usual general results such as Holley [8] implicitly assume flip rates are strictly positive and so do not apply to the East process. So even (2) may be a new result. Note also that the East process is not *monotone* (or *attractive*) in the sense of interacting particle systems [11]. Our *extension coupling* in section 3.1 uses a weaker structural property.

4. For any stationary reversible Markov process $\mathbf{X}(t)$, there are two useful ways of viewing the relaxation time τ. One is via the "infinitesimal time" Rayleigh-Ritz variational characterization (e.g. [2] eq. 6.2.10):

$$\tau = \sup_g \frac{\text{var } g}{\mathcal{E}(g,g)} \tag{3}$$

where var g is the variance of $g(\mathbf{X}(0))$; and

$$\mathcal{E}(g,g) = \tfrac{1}{2} \lim_{t \downarrow 0} t^{-1} E(g(\mathbf{X}(t)) - g(\mathbf{X}(0)))^2.$$

The other is via the finite-time maximal correlation property:

$$\sup_{f,g} \text{cor}(f(\mathbf{X}(0)), g(\mathbf{X}(t)) = \exp(-t/\tau), \quad 0 \le t < \infty \tag{4}$$

where cor denotes correlation. In general the *sup*s are over all square-integrable functions; in our setting of interacting particle systems, in (4) we may restrict to bounded functions depending on only finitely many sites. See the Appendix. Our arguments are purely finitistic – we derive estimates of the spectral gap for the East process on states $\{0, 1, \ldots, n\}$ which are uniform in large n – and we downplay the routine abstract arguments needed to pass to the limit infinite-site models.

5. As well as the (mathematically natural) relaxation time $\tau(p)$ of the whole East process, it is physically natural to consider the single-spin asymptotic relaxation time $\tau^0(p)$ defined in terms of the stationary process by

$$P(X_0(t) = 1|X_0(0) = 1) - p = \exp(-\tfrac{t}{\tau^0(p)\pm o(1)}) \text{ as } t \to \infty.$$

One immediately has $\tau^0(p) \le \tau(p)$ and one expects equality, but there is no simple general proof of equality.

6. The procedure for deriving the East process from Glauber dynamics for product measure has a natural abstraction to arbitrary reversible Markov processes. Taking for simplicity finite state space $\mathcal{X}$, let $Q(x, y)$ be the transition rate matrix and π the stationary distribution of a reversible chain, and let S be a symmetric subset of $\mathcal{X} \times \mathcal{X}$. Then the derived chain

$$\hat{Q}(x, y) = Q(x, y)1_{(y \neq x,\ (x,y)\in S)}$$

is reversible with the same stationary distribution π. The variational principle immediately implies that the relaxation time for the derived chain cannot be smaller than for the original chain. Upper bounding such relaxation times, when the original chain is e.g. the usual Glauber dynamics for the Ising model, is a natural open problem.

2 Heuristics

Our proofs are indirect, because we are unable to do sharp calculations directly with the East process. Underlying our proofs is an intuitive visualization of how the process evolves, which we shall outline in this section.

2.1 Visualizing a realization

Fix small p. A typical particle at a typical time is isolated from other particles by a distance of order $1/p$. Figure 1 illustrates schematically a realization of the East process from an isolated particle at site 0.

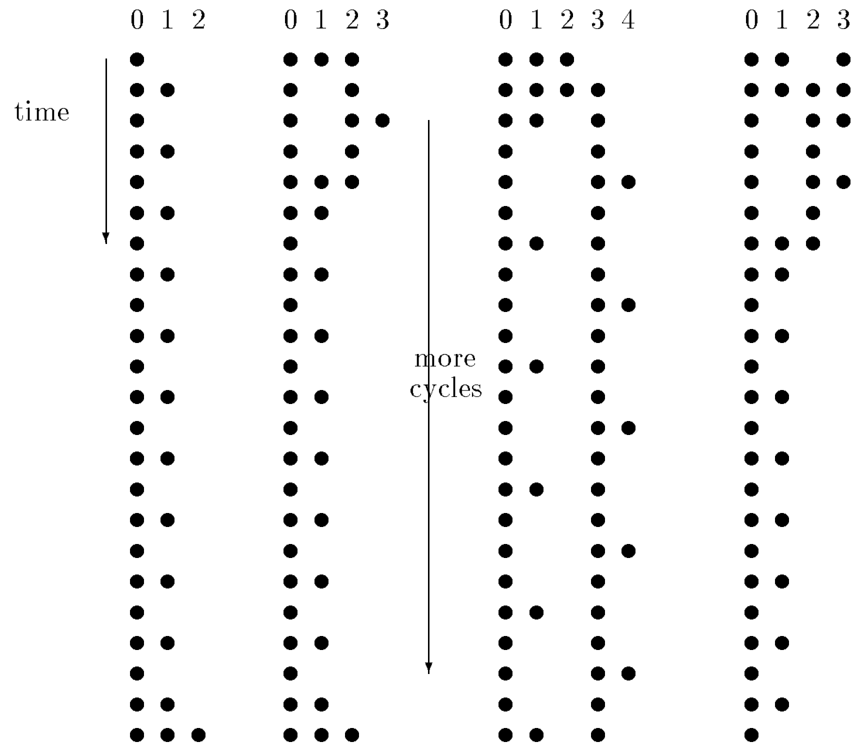

Fig. 1. Schematic representation of successive configurations in the East process started from a particle at site 0. Columns represent consecutive time intervals. From configuration $\bullet\bullet$ the process is much more likely to make a transition to $\bullet$ than to $\bullet\bullet\bullet$; column 1 represents an order $1/p$ number of cycles $\bullet \to \bullet\bullet \to \bullet$ before first attaining $\bullet \bullet \bullet$. From there, it is unlikely to make a transition to $\bullet \bullet \bullet\bullet$ but instead is equally likely to transition to $\bullet\bullet$ or to $\bullet \circ \bullet$. Column 2 indicates that either of the latter options will likely cause the process to return to $\bullet$. It therefore requires order $1/p$ more cycles $\bullet \to \bullet \bullet \bullet \to \bullet$ before first attaining $\bullet \bullet \bullet \ \bullet$. From there it is possible to reach a locally recurrent configuration $\bullet \circ \circ \bullet$ (column 3) but the process will subsequently return to $\bullet \bullet \bullet \ \bullet$ and the other likely possibilities lead (column 4) back to $\bullet$.

The moral of figure 1 is that, starting from an isolated particle, it takes a long time to reach a configuration with another particle at some distance $1 \ll d \ll 1/p$ from the original particle; and such configurations tend to relapse back to the original configuration.

2.2 Minimum-energy paths

For this section, take the site-space to be $\mathbb{Z}^+$ but consider only finite configurations. Write $|\mathbf{x}| := \sum_i x_i$ for the number of particles in configuration $\mathbf{x}$. A natural metric (in fact, an ultrametric) on configuration space is

$$h(\mathbf{x}, \mathbf{x}') := \min_{\text{paths } (\mathbf{x}^j) \text{ from } \mathbf{x} \text{ to } \mathbf{x}'} \max_j |\mathbf{x}^j| \tag{5}$$

where the minimum is over paths $\mathbf{x} = \mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}'$ such that each successive pair $(\mathbf{x}^j, \mathbf{x}^{j+1})$ is a possible transition of the East process. This is natural because, in the usual statistical physics picture, the relative probabilities of configurations $\mathbf{x}$ depend only on $|\mathbf{x}|$, which we can interpret as determining the *energy* of $\mathbf{x}$, so that $h(\mathbf{x}, \mathbf{x}')$ is the (absolute) height of the "energy barrier" separating the configurations. Note that, in contrast to most interacting particle models, if $\mathbf{x}'$ differs from $\mathbf{x}$ only by virtue of moving an isolated particle to an adjacent site, then $\mathbf{x}$ and $\mathbf{x}'$ are *not* close in the h metric; there is no short or low-energy path between them.

Write $\delta_0 + \delta_i + \ldots$ for the configuration consisting of particles at sites 0 and i and $\ldots$. Figure 2 illustrates a recursive construction, which easily proves the following Lemma. This kind of result has certainly been noted in the physics literature, e.g [13] section 4, and [18] show there is in fact equality in (6). Chung et al [3] give further analysis of such combinatorics.

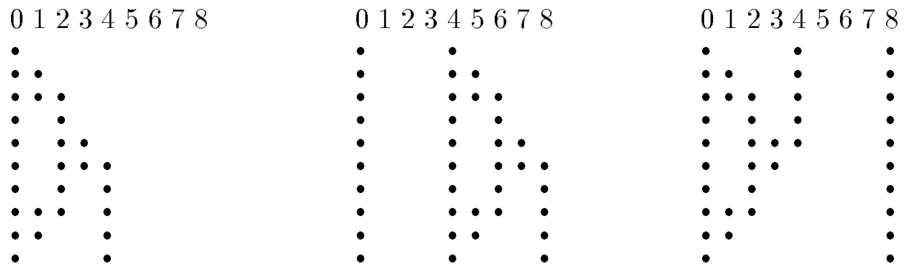

Fig. 2. A path from δ_0 to $\delta_0 + \delta_4$ (left column) is used to define a path from δ_0 to $\delta_0 + \delta_8$. The latter path has length 3 times the former length, and uses one extra particle.

Lemma 2 *Let $m \geq 0$. There exists a path from configuration δ_0 to configuration $\delta_0 + \delta_{2^m}$ of length 3^m such that the maximum of $|\mathbf{x}|$ over configurations $\mathbf{x}$ on the path equals $m + 2$. In particular*

$$h(\delta_0, \delta_0 + \delta_{2^m}) \leq m + 2. \tag{6}$$

2.3 Heuristic analysis of $\tau(p)$

A very crude heuristic analysis (which may be regarded as a summary of the non-rigorous argument in [18]) involves the following two ideas.
(i) Write $s(p,j)$ for the mean time for the East process started at δ_0 to reach some configuration with site j occupied. Since particles are typically separated by distance $1/p$, the relaxation time should be roughly $s(p,1/p)$, since this is the time until a particle has some influence on its initial nearest neighbor particle.
(ii) Lemma 2 suggests that getting from configuration δ_0 to a configuration with a particle at site j involves an "excursion" going over an energy barrier of height roughly $h = \log_2 j$, in other words through configurations of probability roughly p^h. An excursion required to pass through states of some exponentially small probability q should take time roughly $1/q$. So $s(p,j)$ should be roughly $(1/p)^h$.

Combining (i) and (ii) suggests $\tau(p) \approx s(p,1/p) \approx (1/p)^{\log_2 1/p}$, which is (1).

2.4 A continuum limit process?

We would like to prove the following conjecture (in which the $t(p)$ would be, up to constant factors, the relaxation times $\tau(p)$) concerning the extent of the "excursions" above.

Conjecture 3 *Let $\mathbf{X}^0(t)$ be the East process on sites $\mathbb{Z}^+$, started from a single particle at site* 0. *Let $r(\mathbf{x})$ be the rightmost occupied site in configuration $\mathbf{x}$. Then there exist constants $t(p)$ and a function $G(x), 0 < x < \infty$ such that, for all $\omega(p) \downarrow 0$ sufficiently slowly,*

$$P\left(\sup_{0 \le s \le \omega(p)t(p)} r(\mathbf{X}^0(s)) > \tfrac{x}{p}\right) \sim G(x)\omega(p) \textit{ as } p \downarrow 0 \textit{ with } x \textit{ fixed.} \qquad (7)$$

If this were true with suitable tail conditions on $G(\cdot)$, it would follow quite easily (using the qualitative properties of section 3.1) that as $p \downarrow 0$ the stationary East process on $\mathbb{Z}$, rescaled by taking sites to be distance p apart and speeding up time by $t(p)$, converges to a limit process $(\Xi(t), 0 \le t < \infty)$ described as follows.
(a) At fixed time t, $\Xi(t)$ is a Poisson (rate 1) process of particles on R.
(b) Each particle creates "waves" of random lengths, the rate of creation of waves of length $> l$ being $G(l)$.

(c) A wave $(x, x+l]$ instantaneously deletes all particles in $(x, x+l]$ and replaces them by a Poisson (rate 1) process of particles on $(x, x+l]$.

But proving the conjecture seems difficult; we do not have even a heuristic derivation of a formula for $G(x)$.

3 The wave process

In this section we take the site space to be $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$. We generalize the East process by introducing, in addition to $0 < p < 1$, a second parameter $v \in \{1, 2, 3, \ldots\}$, and prescribing transition rates as follows.

> Each particle (at site i, say) at rate 1 creates a "wave" of length v, which instantaneously deletes any particles in sites $\{i+1, i+2, \ldots, i+v\}$ and replaces them by an i.i.d. Bernoulli(p) process of particles on sites $\{i+1, i+2, \ldots, i+v\}$.

Also specify that site 0 is always occupied. We call this the *wave process*, and write it as $\mathbf{W}(t) = (W_i(t),\ i \in \mathbb{Z}^+)$. The particular case $v = 1$ is the East process. It is easy to check that the general wave process is reversible and its stationary distribution is i.i.d. Bernoulli(p) on sites $\{1, 2, \ldots\}$ with the fixed particle at site 0.

In section 3.1 we give some qualitative properties which do not depend on v; then in section 3.2 we give a quantitative bound on the relaxation time for large v.

If I is an interval of sites then we write $\mathbf{W}_I(t)$ for the restriction to sites I of $\mathbf{W}(t)$.

3.1 Qualitative properties of the wave process

The first two properties are intuitively obvious from the definitions; we will not spell out details.

Consistency. One could define the wave process on a finite site space $\{0, 1, 2, \ldots, i_0\}$. These processes are consistent as i_0 varies. In other words, taking first the wave process $\mathbf{W}$ on sites $\mathbb{Z}^+$, the restricted process $\mathbf{W}_{[0,i_0]}$ is distributed as the wave process on sites $\{0, 1, 2, \ldots, i_0\}$.

Conditional independence. In the setting above, condition on the entire restricted process $(\mathbf{W}_{[0,i_0]}(t), 0 \leq t < \infty)$ and on the times σ_j and right endpoints r_j of waves emanating from sites in $[0, i_0]$ which have $r_j > i_0$. Then conditionally, the process $\mathbf{W}_{[i_0+1,\infty)}(t)$ evolves as the wave process, except that at times σ_j the sites in $[i_0+1, r_j]$ are reset to i.i.d. Bernoulli(p).

We next spell out a slightly more subtle *conditional stationarity* property. Consider the wave process started at time 0 with an arbitrary initial configuration $\mathbf{x}(0)$, and with an arbitrary initial particle distinguished. At each subsequent time exactly one particle is distinguished, according to the following rule. Once distinguished, a particle (at site j say) remains distinguished until the first time some wave emanates from some site $i \in \{j-v, j_v+1, \ldots, j-1\}$ and at that time the particle at site i becomes the distinguished particle.

Let $q(t)$ be the position of the distinguished particle at time t.

Lemma 4 *Fix an interval of sites $I = [i_0, i_1]$. Suppose the initial random configuration $\mathbf{W}(0)$ has a particle at site i_0, the distinguished particle, and has i.i.d. Bernoulli(p) distribution on $(i_0, i_1]$. For each t, conditional on $\{q(t) = i'\}$, the distribution of $\mathbf{W}_{[i'+1,i_1]}(t)$ (that is, of the wave process at time t on sites $\{i'+1, i'+2, \ldots, i_1\}$) is i.i.d. Bernoulli(p).*

Proof. Let $0 < U_1 < U_2 < \ldots$ be the times when the distinguished particle changes. Let $\mathbf{W}(U_j-)$ and $\mathbf{W}(U_j)$ be the configurations before and after the change. Inductively suppose that conditional on U_j and on $\{q(U_j) = i'\}$, the distribution of $\mathbf{W}_{[i'+1,i_1]}(U_j)$ is i.i.d. Bernoulli(p). And site i' is occupied at time U_j. It follows from the basic stationarity property (of the wave process on $\mathbb{Z}^+$, translated to $[i', \infty)$) that in the absence of waves crossing into $[i', \infty)$ from below, the distribution at any subsequent time t of $\mathbf{W}_{[i'+1,i_1]}(t)$ will be i.i.d. Bernoulli(p). By the conditional independence property, the distribution of $\mathbf{W}_{[i'+1,i_1]}(U_{j+1}-)$ is also i.i.d. Bernoulli(p), and this remains true conditionally on U_{j+1}. Given $q(U_{j+1}) = i'' < i'$, the configuration $\mathbf{W}_{[i''+1,i_1]}(U_{j+1})$ consists of the existing configuration $\mathbf{W}_{[r_j+1,i_1]}(U_{j+1}-)$ (which is i.i.d. Bernoulli(p)), together with the new configuration on $[i''+1, r_j]$ created by the wave, which is also i.i.d. Bernoulli(p), thus making the whole configuration $\mathbf{W}_{[i''+1,i_1]}(U_{j+1})$ have i.i.d. Bernoulli(p) distribution. This carries the induction forward, and establishes the lemma. ■

We next give a coupling construction. Let $\mathbf{x}^0$ and $\mathbf{x}^1$ be configurations on $\mathbb{Z}^+$. Say $\mathbf{x}^1$ is an *extension* of $\mathbf{x}^0$ if $\mathbf{x}^0$ has only a finite number of occupied sites and if the two configurations coincide on $[0, r]$, where $r = r(\mathbf{x}^0)$ is the position of the rightmost particle of $\mathbf{x}^0$. In other words, $\mathbf{x}^1$ consists of the particles in $\mathbf{x}^0$ and (perhaps) extra particles at arbitrary positions greater than r.

Lemma 5 *Suppose $\mathbf{x}^1$ is an extension of $\mathbf{x}^0$. Then there exists a coupling $((\mathbf{W}^0(t), \mathbf{W}^1(t)), 0 \le t < \infty)$ of the wave processes with initial configurations*

$\mathbf{x}^0$ *and* $\mathbf{x}^1$ *such that at each time* t*, the configurations* $\mathbf{W}^1(t)$ *and* $\mathbf{W}^0(t)$ *coincide on sites* $[0, \sup_{0 \le s \le t} r(\mathbf{W}^0(s))]$.

Call this the *extension coupling.*

Proof. Suppose $\mathbf{x}^1$ is an extension of $\mathbf{x}^0$. Then we can couple transitions of the joint process from these configurations by specifying that when a wave emerges from a particle (site i, say) of $\mathbf{x}^1$ and creates new particles, then if site i contains a particle of $\mathbf{x}^0$ we copy the wave and the positions of new particles in the other process, and otherwise do nothing. This clearly maintains the "extension" property. Furthermore, if $\mathbf{x}^0$ and $\mathbf{x}^1$ coincide on sites $[0, r^*]$ for some $r^* \ge r(\mathbf{x}^0)$ then the two coupled processes will always agree on that interval. ■

Now let $(\mathbf{W}^0(t), 0 \le t < \infty)$ be the wave process on $\mathbb{Z}^+$ started with only one particle at site 0. Let $R_t = r(\mathbf{W}^0(t))$ be the site of the rightmost particle of $\mathbf{W}^0(t)$.

Lemma 6 *Suppose there exists* $\lambda > 0$ *such that for each* i_0

$$P(\sup_{0 \le s \le t} R_s \le i_0) = O(e^{-\lambda t}) \text{ as } t \to \infty. \tag{8}$$

Then the wave process on $\mathbb{Z}^+$ *has spectral gap at least* λ.

Proof. Fix i_0 and write $I = [0, i_0]$. Let $\mathbf{W}^0(t)$ be as above, let $\mathbf{W}(t)$ be the stationary wave process and let $\mathbf{W}^*(t)$ be the wave process started with arbitrary initial configuration $\mathbf{x}^*$. By Lemma 5 we can couple $\mathbf{W}^0$ and $\mathbf{W}$ such that

$$P(\mathbf{W}^0_I(t) \ne \mathbf{W}_I(t)) \le P(\sup_{0 \le s \le t} R_s < i_0).$$

Similarly we can couple $\mathbf{W}^0$ and $\mathbf{W}^*$ such that

$$P(\mathbf{W}^0_I(t) \ne \mathbf{W}^*_I(t)) \le P(\sup_{0 \le s \le t} R_s < i_0).$$

So, writing $||\cdot||$ for variation distance,

$$\sup_{\mathbf{x}} ||P(\mathbf{W}_I(t) \in \cdot | \mathbf{W}_I(0) = \mathbf{x}) \; - \; \pi_I(\cdot)|| \le 2P(\sup_{0 \le s \le t} R_s < i_0) \tag{9}$$

where π_I is i.i.d. Benoulli(p) distribution on I. In a finite-state reversible chain, the spectral gap equals the exponent in the asymptotic rate of convergence to stationarity, and so (9) and (8) imply that the restricted chain $\mathbf{W}_{[0,i_0]}$ has spectral gap at least λ. Since this bound is uniform in i_0 it extends by consistency to the wave process on sites $\mathbb{Z}^+$ (see Appendix for more details).

3.2 The supermartingale analysis

Here is the main result of section 3.

Proposition 7 *If $v = v(p) > \frac{10}{p}+2$ then the spectral gap of the wave process on $\mathbb{Z}^+$ is at least $3/10$, for sufficiently small p.*

In outline, the idea is to apply Lemma 6. To prove (8) for some given λ it would suffice to show that for some $\theta > 0$

$$\exp(\lambda t - \theta R_t) \text{ is a supermartingale.} \tag{10}$$

For then $E \exp(-\theta R_t) \leq e^{-\lambda t}$ and so $P(R_t \leq x) \leq e^{\theta x} e^{-\lambda t}$. Unfortunately (10) cannot be exactly true, because there are "bad" configurations from which R_t tends to decrease rather than increase (for instance, if sites $R_t - v$ through $R_t - v + 100$ are occupied but sites $R_t - v + 101$ through $R_t - 1$ are unoccupied). So we use a more elaborate argument which finesses bad configurations by establishing the supermartingale property only at embedded random times (Lemma 8). Incidently, one could modify the wave process by allowing the waves to have *random* length V, and taking V to have geometric distribution with sufficiently large mean the supermartingale property (10) would be easy to verify. However, using unbounded V would make the comparison argument in section 4 become more complicated.

Proof of Proposition 7. Write $v_0 = \lceil 5/p \rceil$, so $v \geq 2v_0$. We first consider the wave process on $\mathbb{Z}^+$, started at time 0 with some arbitrary finite configuration of particles. Write R_0 for the position of the rightmost particle at time 0. Let U be the first time that either a wave emerges from the particle at R_0 or the particle at R_0 is removed by a wave emanating from another particle. Let $T \geq U$ be the time of the first wave whose right end is $\geq R_0 + v_0$. Note that $T \neq U$ only in the case where the particle at R_0 is removed by a wave whose rightmost limit is between R_0 and $R_0 + v_0$; this is the case whose analysis is more difficult. Fix some $1 > \lambda > 0$ and define

$$M_t = \exp(\lambda t - \tfrac{p}{2} R_t).$$

Lemma 8 *Let p be sufficiently small. For any finite initial configuration,*

$$E M_{T \wedge (U+1)} \leq \frac{e^{\lambda - 1}}{1 - \lambda} M_0, \quad 0 < \lambda < 1.$$

Here $T \wedge (U+1) := \min(T, U+1)$. The proof has three parts. At (12) we show that $T \leq U + 1$ is the likely alternative. At (11) we show that R_T

tends to be larger than R_0 on $\{T \leq U+1\}$. At (14) we show that R_T will not be much larger than R_0 on $\{T > U+1\}$.

Proof of Lemma 8. We first argue

$$E\left(\exp(-\tfrac{p}{2}(R_T - R_0))|T,U\right) \leq 2e^{-5/2} + o(1) \text{ on } \{T \leq U+1\} \tag{11}$$

where $o(1)$ denotes a constant tending to 0 as $p \downarrow 0$. At time T there is some wave with right endpoint y' and R_T is stochastically larger than $y'+1-\eta$, where η has geometric(p) distribution and is independent of T, U. Since $y' \geq R_0 + v_0$ we see that on $\{T < \infty\}$

$$\begin{aligned} E\left(\exp(-\tfrac{p}{2}(R_T - R_0))|T,U\right) &\leq E\exp(-\tfrac{p}{2}(v_0 + 1 - \eta)) \\ &\leq e^{-5/2} E\exp(-p\eta/2) \\ &= e^{-5/2}(2 + o(1)). \end{aligned}$$

We next shall argue that for sufficiently small p

$$P(T > U+1) \leq 1/100. \tag{12}$$

(This will in fact to proved and used in a certain "conditional" form). If $R_0 \leq v_0$ then $P(T \neq U) = 0$ by the note above Lemma 8, because the rightmost end of any wave emanating from below R_0 must be $\geq v \geq R_0 + v_0$. So assume $R_0 > v_0$. Condition on the restriction to sites $[0, R_0 - v_0]$ of the process $(\mathbf{W}(t), 0 \leq t < \infty)$, and on the times and rightmost ends (t_*, x_*) of waves ending at positions to the right of $R_0 - v_0$. (This conditioning is denoted by "waves" in (13) below). In order that $T > U+1$ there must

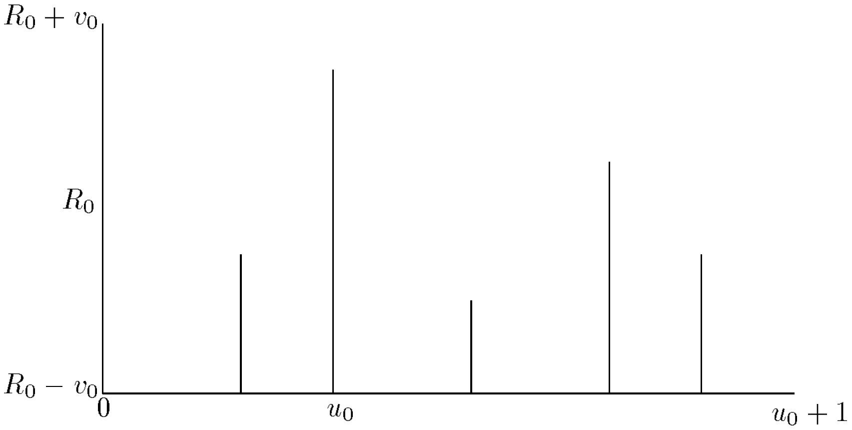

Fig. 3. Waves.

be some first wave, at time u_0, with rightmost end $x_0 \in [R_0, R_0 + v_0)$, and all the other rightmost ends of waves at times before $u_0 + 1$ must also be $< R_0 + v_0$. Figure 3 illustrates such waves.

By the conditional independence property (section 3.1) the restriction of the wave process to the sites $I := (R_0 - v_0, R_0 + v_0]$ evolves according to the usual wave process rules, except that the waves (on which we are conditioning) reset the sites they cover to i.i.d. Bernoulli(p). The conditional probability that $T > U + 1$ is the probability that no particle in sites I (neither an initial particle nor a particle created by any of the conditioning waves) has a wave emanating from it before time $u_0 + 1$ (because any such wave would have rightmost end greater than $R_0 - v_0 + v \geq R_0 + v_0$). We can upper bound this conditional probability by considering only particles created by the wave at time u_0 and subsequent waves, and only sites $I^* := (R_0 - v_0, R_0]$. For each site $i \in I^*$ we can decompose the time interval $[u_0, u_0 + 1]$ into subintervals J starting at the successive times when a wave meets site i. let $\mathcal{J}$ denote the set of such intervals J obtained by varying i, and let $(\beta_J, J \in \mathcal{J})$ be the i.i.d. Bernoulli(p) random variables indicating whether the wave created a particle at site i. The number of waves created by all these particles is Poisson with conditional mean $M := \sum_J |J|\beta_J$, where $|J|$ is the length of interval J. So

$$P(T > U + 1 | \text{ waves}) \leq E(\exp(-M)|\mathcal{J}). \tag{13}$$

Now the constraints on the family $\mathcal{J}$ are

$$|J| \leq 1 \ \forall J; \quad \sum_J |J| = v_0.$$

A routine convexity argument shows that, subject to these constraints, $E(\exp(-M)|\mathcal{J})$ is maximized in the case where each $|J| = 1$. So

$$E(\exp(-M)|\mathcal{J}) \leq E \exp(-B(v_0, p))$$

where $B(\cdot, \cdot)$ has Binomial distribution. As $p \to 0$ we have $v_0 p \to 5$ and so

$$E \exp(-B(v_0, p)) \to \exp(-5(1 - e^{-5})) < 1/100$$

giving (12).

We digress to record an elementary calculation

Lemma 9 *If Y is stochastically smaller than exponential(1) then for any event D*

$$E\exp(Y/2)1_D \leq 2\sqrt{P(D)}.$$

Proof. It is enough to verify the case where Y has exponential(1) distribution and D is of the form $[x,\infty)$. But in that case the asserted inequality becomes

$$\int_x^\infty e^{y/2}\ dy \leq 2e^{-x/2}$$

which is true with equality. ■

At time U there is some wave $[x,x+v]$ with $x+v \geq R_0$. By taking time U and interval $[x,x+v]$ as the initial time and interval in Lemma 4, the conclusion of that lemma easily implies that $R_0 - R_{U+1} + 1$ is stochastically smaller than geometric(p). This implies $p(R_0 - R_{U+1})$ is stochastically smaller than exponential(1). So by applying Lemma 9 to $Y := p(R_0 - R_{U+1})$ and $D := \{T > U+1\}$ gives, using (12),

$$E\left(\exp(-\tfrac{p}{2}(R_{U+1}-R_0))1_{(T>U+1)}|U\right) \leq 1/5. \qquad (14)$$

Now split the quantity under study in Lemma 8 over the events $\{T \leq U+1\}$ and $\{T > U+1\}$:

$$\begin{aligned} e^{\frac{p}{2}R_0}\ EM_{T\wedge(U+1)} &= Ee^{\lambda T}\exp(-\tfrac{p}{2}(R_T-R_0))1_{(T\leq U+1)} \\ &+ Ee^{\lambda(U+1)}\exp(-\tfrac{p}{2}(R_{U+1}-R_0))1_{(T>U+1)}. \end{aligned}$$

Consider the first term. By conditioning on T,U and using (11)

$$(\text{first term}) \leq (2e^{-5/2}+o(1))Ee^{\lambda T}1_{(T\leq U+1)} \leq (2e^{-5/2}+o(1))Ee^{\lambda(U+1)}.$$

Similarly, by conditioning on U and using (14),

$$(\text{second term}) \leq \tfrac{1}{5}Ee^{\lambda(U+1)}.$$

Combining these two bounds,

$$e^{\frac{p}{2}R_0}\ EM_{T\wedge(U+1)} \leq (2e^{-5/2}+\tfrac{1}{5}+o(1))e^{\lambda}Ee^{\lambda U}.$$

The first term works out numerically to be $< e^{-1}$. And U is stochastically smaller than the exponential(1) time at which a wave would emanate from the initial particle at R_0, so $Ee^{\lambda U} \leq 1/(1-\lambda)$. This establishes Lemma 8. ■

Returning to the proof of Proposition 7, choose $\lambda = 3/7$ to make $\frac{e^{\lambda-1}}{1-\lambda} < 1$. Consider $(\mathbf{W}^0(t), 0 \leq t < \infty)$, the wave process on $\mathbb{Z}^+$ started with only one particle at site 0. Define stopping times $0 = S_0 < S_1 < S_2 < \ldots$ by:

$S_{k+1} - S_k$ is the time $T \wedge (U+1)$ defined above Lemma 8, applied to the wave process $(\mathbf{W}^0(S_k + t),\ 0 \leq t < \infty)$.

So Lemma 8 implies that $(M_{S_k}, 0 \leq k < \infty)$ is a supermartingale. Fix t and define

$$\kappa := \min\{k : S_k \geq at\}$$

where the constant $0 < a < 1$ will be specified later. Now

$$\begin{aligned} 1 &\geq EM_{S_\kappa} \quad \text{(optional sampling theorem)} \\ &\geq EM_{S_\kappa} 1_{(R(S_\kappa) \leq x)} \\ &\geq \exp(\lambda at - px/2) P(R(S_\kappa) \leq x) \end{aligned}$$

implying

$$P(R_{S_\kappa} \leq x) \leq e^{px/2} e^{-\lambda at}. \tag{15}$$

We next need to bound the "overshoot" $S_\kappa - at$. Lemma 10 below implies $P(S_\kappa - at \geq 1 + u) \leq e^{-u},\ 0 \leq u < \infty$ and hence

$$P(S_\kappa > t) \leq e \cdot e^{-(1-a)t}. \tag{16}$$

Finally,

$$\begin{aligned} P\left(\sup_{0 \leq s \leq t} R_s \leq x\right) &\leq P(R_{S_\kappa} \leq x) + P(S_\kappa > t) \\ &\leq 2e^{1 \wedge px/2} e^{-\max(\lambda a, (1-a))t} \text{ by } (15, 16). \end{aligned}$$

Having specified $\lambda = 3/7$ we now specify $a = 7/10$ and the bound is $O(e^{-3t/10})$ as $t \to \infty$. So Proposition 7 follows from Lemma 6.

Lemma 10 *$S_\kappa - at$ is stochastically smaller than $1 + \eta$, where η denotes an exponential(1) r.v.*

Proof. Write $t_0 = at$, so that $\kappa = \min\{k : S_k \geq t_0\}$. Regard time t_0 as the present, and condition on the past $(\mathbf{W}^0(t), 0 \leq t \leq t_0)$. The conditioning tells us the time $S_{\kappa-1}$ and the position of the rightmost particle $R_0 = r(S_{\kappa-1})$ at that time. Resetting time to restart at time $S_{\kappa-1}$, we are in the setting of figure 3, with initial configuration $\mathbf{W}^0(S_{\kappa-1})$. Now $S_\kappa - S_{\kappa-1} = T \wedge (U+1)$; we see from the conditioning that $T \wedge (U+1) > t_1 := t_0 - S_{\kappa-1}$ (the present time is now t_1) and we are interested in the distribution of the overshoot $\zeta := (T \wedge (U+1)) - t_1$. If $U \leq t_1$ then $\zeta \leq 1$ and we are done, so suppose

$U > t_1$. In that case the rightmost particle at time t_1 is still the particle at site R_0, and the future waiting time until U is at most the exponential time η until a wave would emanate from the particle at R_0. So $\zeta \leq 1 + \eta$ and we are done. ■

Remark. Lemma 10 is slightly subtle; the fact that the conditional distributions of $S_k - S_{k-1}$ are stochastically smaller than $1 + \eta$ is not enough to get a bound on overshoots.

3.3 The wave process on $\mathbb{Z}$

Going from the wave process on $\mathbb{Z}^+$ to the wave process on $\mathbb{Z}$ involves some easy arguments which we shall just outline. Consider first a branching random walk $\mathbf{B}(t)$, in which each site may have more than one particle, and particles independently at rate 1 create a wave of offspring, one at each of the v sites to the right of the parent particle, with particles never being killed. It is well known that (in discrete time, starting from a single particle) the position $r(\mathbf{B}(t))$ of the rightmost occupied site grows asymptotically at a finite linear rate [1], and the same arguments give the essentially weaker conclusion of the next lemma.

Lemma 11 *For the branching random walk $\mathbf{B}(t)$ where initially all sites in $(-\infty, 0)$ are occupied by a single particle,*

$$\lim_{L \to \infty} P(r(\mathbf{B}(t)) > L) = 0, \quad t \text{ fixed.}$$

Next, observe there is a "basic coupling" of two versions of the wave process, as follows. If a site i is occupied in each version, make the wave-times from i and subsequent replacements be identical; for sites which are unmatched (occupied in one version only) let the waves occur independently. It is easy to check the following.

Lemma 12 *Given two initial configurations for the wave process $\mathbf{W}^{(1)}(0)$ and $\mathbf{W}^{(2)}(0)$, let $\mathbf{B}(0)$ be the set of unmatched sites. Then the basic coupling $(\mathbf{W}^1(t), \mathbf{W}^2(t))$ can be constructed jointly with the branching random walk $\mathbf{B}(t)$ such that for each t the set of unmatched sites $(\mathbf{W}^1(t), \mathbf{W}^2(t))$ is a subset of the set of occupied sites in $\mathbf{B}(t)$.*

Now write $\mathbf{W}(t)$ for the stationary wave process on sites $\mathbb{Z}^+$. Let σ_L be the shift map taking $(x_i, i \geq 0)$ to $(x_{i-L}, i \geq 0)$.

Lemma 13 *As $L \to \infty$ the processes $\sigma_L(\mathbf{W}(t))$ converge weakly to a process $\widetilde{\mathbf{W}}(t)$, the stationary wave process on sites $\mathbb{Z}$.*

Proof. It is enough to show that we can couple $\sigma_{L_1}(\mathbf{W}(t))$ and $\sigma_{L_2}(\mathbf{W}(t))$ such that, for fixed t and i_0, as

$$L_1 \to \infty,\ L_2 \to \infty,\ L_1 < L_2$$

we have

$$P(\sigma_{L_1}(\mathbf{W}(t))|_i = \sigma_{L_2}(\mathbf{W}(t))|_i \ \forall i \geq i_0) \to 1. \tag{17}$$

Use the basic coupling above, where the initial configurations coincide except on sites $[-L_2, -L_1)$, and Lemma 12 to show that the probability in (17) is at least

$$P(r(\mathbf{B}(t)) < i_0)$$

where $\mathbf{B}(0)$ has no particles outside $[-L_2, -L_1)$. Use Lemma 11 to show this probability $\to 1$ as $L_1 \to \infty$. ∎

From the "maximal correlation" interpretation (4) of relaxation time, using functionals depending on only finitely many sites, one sees that for a weakly convergent sequence of interacting particle systems, the relaxation time of the limit is at most the limit of the relaxation times. So the bound from Proposition 7 goes through to the limit in Lemma 13:

Corollary 14 *If $v = v(p) > \frac{10}{p} + 2$ then the spectral gap of the wave process on $\mathbb{Z}$ is at least $3/10$, for sufficiently small p.*

4 The comparison argument

4.1 The general inequality

Proposition 15 states the general inequality we use, in our setting of the East process $\mathbf{X}$ and the wave process $\mathbf{W}$ (on state space $\mathbb{Z}$, with the same parameter p). Readers unfamiliar with this kind of comparison argument should consult [4], where examples in the less technical finite-space setting are explained carefully.

Let $Q^{\mathbf{X}}(\cdot)$ be the equilibrium flow measure on the space $\mathcal{X}$ of possible transitions $(\mathbf{x}, \mathbf{x}')$ of $\mathbf{X}$. That is, the first marginal of $Q^{\mathbf{X}}$ is the stationary, Bernoulli(p), law and the conditional law is the transition rate. Let $Q^{\mathbf{W}}(\cdot)$ be the equilibrium flow measure for $\mathbf{W}$. For each possible transition $(\mathbf{w}, \mathbf{w}')$ of $\mathbf{W}$, define a path $\mathbf{w} = \mathbf{x}(0), \mathbf{x}(1), \ldots, \mathbf{x}(l) = \mathbf{w}'$ whose steps $(\mathbf{x}(i-1), \mathbf{x}(i))$ are possible transitions of $\mathbf{X}$, and write $N_{(\mathbf{w},\mathbf{w}')}(\cdot)$ for the counting measure on $\mathcal{X}$ which counts the transitions $(\mathbf{x}(i-1), \mathbf{x}(i))$. Then define a measure on $\mathcal{X}$ by:

$$\widetilde{Q}(\cdot) := \int N_{(\mathbf{w},\mathbf{w}')}(\cdot)\ Q^{\mathbf{W}}(d\mathbf{w}, d\mathbf{w}'). \tag{18}$$

Proposition 15 *Suppose we can choose paths such that, for constants $B, L < \infty$,*
(i) each path length is at most L
(ii) the density $d\widetilde{Q}/dQ^{\mathbf{X}}$ is bounded a.e. by B (see Remark below).
Then the relaxation times of the two processes satisfy

$$\tau(\mathbf{X}) \leq BL\ \tau(\mathbf{W}).$$

Diaconis and Saloff-Coste [4] Theorem 2.1 prove this in the discrete time, finite state space setting, but since the argument rests only on the general variational characterization (3) and a Cauchy-Schwarz bound, it extends to our setting without essential alteration. See Lemmas 1.13 – 1.17 of Holley [8].

Remark. If we were on a finite state space then (18) could be written as a sum

$$\widetilde{Q}(\mathbf{x}, \mathbf{x}') := \sum_{(\mathbf{w},\mathbf{w}')} N_{(\mathbf{w},\mathbf{w}')}(\mathbf{x}, \mathbf{x}')\ Q^{\mathbf{W}}(\mathbf{w}, \mathbf{w}').$$

One compares this quantity with $Q^{\mathbf{X}}(\mathbf{x}, \mathbf{x}')$ by considering the ratio

$$\frac{\widetilde{Q}(\mathbf{x}, \mathbf{x}')}{Q^{\mathbf{X}}(\mathbf{x}, \mathbf{x}')} = \frac{d\widetilde{Q}}{dQ^{\mathbf{X}}}(\mathbf{x}, \mathbf{x}') \text{ say.}$$

The Radon-Nikodym density in (ii) is the analogous function $\frac{d\widetilde{Q}}{dQ^{\mathbf{X}}}(\mathbf{x}, \mathbf{x}')$ in the infinite-site setting.

4.2 Applying the comparison inequality

The "distinguished paths" required to implement the comparison method are readily constructed in terms of the "minimum energy" paths of Lemma 2. At many places the bounds are crude.

Lemma 16 *Let $m \geq 1$. Take site space $[0, 2^m]$ with site 0 always occupied. Let $\mathbf{W}$ and $\mathbf{W}'$ be independent with Bernoulli(p) distribution on sites $[1, 2^m]$. Then we can construct a path from $\mathbf{W}$ to $\mathbf{W}'$ of length $2 \cdot 3^m$, using only possible transitions of the East process, such that for each configuration $\mathbf{x}$*

$$E(\textit{number of exits of path from } \mathbf{x}) \leq 2 \cdot 3^m\ p^{|\mathbf{x}|-m-2}. \tag{19}$$

Proof. Write $(\hat{\mathbf{x}}(u), 0 \leq u \leq 3^m)$ for the path from configuration δ_0 to configuration $\delta_0 + \delta_{2^m}$ given in Lemma 2. Given an arbitrary configuration $\mathbf{w}$ with 0 occupied, for each $i \in [0, 2^m]$ write $s_i = \min\{u \geq 0 : \hat{x}_i(u) = 1\}$ and set

$$x_i(u) = \hat{x}_i(u) + 1_{(w_i=1)} 1_{(u<s_i)}.$$

As illustrated in figure 4, this constructs a path $(\mathbf{x}(u))$ of length 3^m from $\mathbf{w}$ to $\delta_0 + \delta_{2^m}$; sites initially occupied in $\mathbf{w}$ remain occupied until the $\hat{\mathbf{x}}$ path first makes the site occupied, but then behave as in the $\hat{\mathbf{x}}$ path. Joining two such paths back-to-back constructs a path of length $2 \cdot 3^m$ between arbitrary configurations $\mathbf{w}$ and $\mathbf{w}'$. It is now enough to show that, for a fixed step u (w.l.o.g. $u \leq 3^m - 1$), when $\mathbf{w}$ has the Bernoulli(p) distribution,

$$P(\mathbf{x}(u) = \mathbf{x}) \leq p^{|\mathbf{x}|-m-2} \quad \forall \mathbf{x}.$$

By Lemma 2 we have $|\hat{\mathbf{x}}(u)| \leq m + 2$. So if $|\mathbf{x}| > m + 2$ then there are at least $|\mathbf{x}| - m - 2$ sites which are occupied in $\mathbf{x}$ but not in $\hat{\mathbf{x}}(u)$; in order for $\mathbf{x}(u) = \mathbf{x}$ it is necessary that all these sites be occupied in the initial $\mathbf{w}$, which has chance $p^{|\mathbf{x}|-m-2}$. ∎

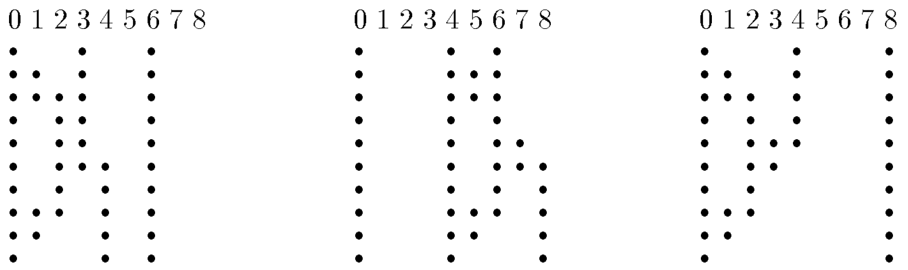

Fig. 4. A path from $\delta_0 + \delta_3 + \delta_6$ to $\delta_0 + \delta_8$.

Return to the setting of Proposition 15, and suppose the wave process has waves of length $v = 2^m$. A possible transition $(\mathbf{w}, \mathbf{w}')$ of the wave process involves a wave from some site i and only affects sites $[i, i + 2^m]$. Use the path from $\mathbf{w}$ to $\mathbf{w}'$ constructed in Lemma 16. So condition (i) in Proposition 15 holds with

$$L = 2 \cdot 3^m \tag{20}$$

We shall argue that condition (ii) holds with

$$B = 2^m \cdot 2 \cdot 3^m \; p^{-m-2}(1-p)^{-2^m}. \tag{21}$$

To argue this, fix some transition $(\mathbf{x}, \mathbf{x}')$ of the East process, and suppose it is site j that flips in this transition. For this to be a step along the distinguished path from $\mathbf{w}$ to $\mathbf{w}'$, the wave involved in the transition $(\mathbf{w}, \mathbf{w}')$ must come from some site $i \in [j - 2^m, j - 1]$. We get an exact expression

$$\frac{d\widetilde{Q}}{dQ^{\mathbf{X}}}(\mathbf{x}, \mathbf{x}') = \sum_{i=j-2^m}^{j-1} 1_{(x_i=1)} \frac{pG(\mathbf{x}_{[i,i+2^m]})}{\pi(\mathbf{x}_{[i,i+2^m]})q} \tag{22}$$

where

- $\mathbf{x}_{[i,i+2^m]}$ denotes $\mathbf{x}$ restricted to sites $[i, i + 2^m]$;
- $\pi(\mathbf{x}_{[i,i+2^m]})$ is its Bernoulli(p) probability;
- $q(= p$ or $1)$ is the transition rate from $\mathbf{x}$ to $\mathbf{x}'$;
- $G(\mathbf{x}_{[i,i+2^m]})$ is the expected number of transitions from $\mathbf{x}_{[i,i+2^m]}$ to $\mathbf{x}'_{[i,i+2^m]}$ in the Lemma 16 path between two configurations which have site i occupied and which are independent Bernoulli(p) on $[i, i + 2^m]$.

Use the inequalities

$$\begin{aligned} q &\geq p \\ \pi(\mathbf{x}_{[i,i+2^m]}) &\geq p^{|\mathbf{x}_{[i,i+2^m]}|}(1-p)^{2^m} \\ G(\mathbf{x}_{[i,i+2^m]}) &\leq 2 \cdot 3^m \; p^{|\mathbf{x}_{[i,i+2^m]}|-m-2} \text{ (Lemma 16)} \\ 1_{(x_i=1)} &\leq 1 \end{aligned}$$

to show that the right side of (22) is bounded by the quantity in (21).

Proof of Theorem 1(a). Combining Proposition 15, with estimates (20,21), with Corollary 14 shows that if p is sufficiently small and

$$2^m \geq \tfrac{10}{p} + 2 \tag{23}$$

then

$$\tau(p) \leq \tfrac{10}{3} \cdot 2^2 \cdot 18^m \; p^{-m-2}(1-p)^{-2^m}.$$

Choosing the smallest $m = m(p)$ which satisfies the constraint (23),

$$\tau(p) \leq \beta(1/p) \; (1/p)^m \text{ as } p \downarrow 0$$

where $\beta(1/p)$ is polynomial in $1/p$. This establishes Theorem 1(a).

5 Proof of the lower bound

5.1 Overview of argument

We shall derive the lower bound by applying the variational characterization (3) to a suitable test function g. In many settings, some "simple and intuitively natural" choice of test function g gives a good lower bound, but here we proceed more indirectly by using a function defined in terms of a different stochastic process, the CRJ process defined below. Heuristically, the CRJ process is designed as a caricature of the East process started from all 1's, where the occupied sites of the CRJ process at time t mimic the sites in the East process which have been occupied *throughout* the time interval $[0, t]$. The idea (cf. section 2.3) is that the time for a particle at site i to be influenced by its nearest-to-the-left particle at site $j < i$ should be $\approx p^{j-i}$. So we envisage "intervals on which the East process has reached local equilibrium", such intervals labeled by the left endpoints, and we envisage intervals merging at rate p^{i-j}. We think of the CRJ an an "approximate dual" of the East process, analogous to the well known exact duality between coalescing random walks and the voter model [11].

Here is the precise definition. Fix p and throughout this section set $n = \lfloor 1/p \rfloor$. Let $S \subseteq \{1, 2, \ldots, n\}$ be a non-empty set of *sites*. Define the *coalescing random jumps* (CRJ) process with initial state S as follows. Initially there is one particle at each site in $S \cup \{0\}$. Each particle in S dies at some random time, determined by the rule

> The rate at which the particle at site i dies at time t equals $p^{D(t,i)}$, where $D(t, i) := \min\{i - j : 0 \leq j < i, \text{ and } j \text{ alive at time } t\}$.

When particle i dies, we shall say it *coalesces* with the particle at the nearest lower-numbered site j. Eventually all particles will coalesce with the particle at site 0. Let $L = L(S)$ be the random site occupied by the last-to-die particle in S. Finally, for a configuration $\mathbf{x} = (x_1, \ldots, x_n\} \in \{0, 1\}^n$ define

$$g(\mathbf{x}) = P(L(S) > n/2), \text{ where } S = \{i : x_i = 1\}$$

setting $g(\mathbf{x}) = 0$ when $\mathbf{x}$ is the zero vector.

It is very easy to estimate the required variance in the variational characterization (3). Since

$$P(g(\mathbf{X}(0)) = 0) \geq P(X_i(0) = 0 \ \forall 1 \leq i \leq n) \to e^{-1} \text{ as } p \downarrow 0$$

and

$$P(g(\mathbf{X}(0))=1) \geq P(X_i(0)=0,\ 1 \leq i \leq n/2;\ X_j(0)=1,\ \text{for some } n/2 < j \leq n)$$

$$\to e^{-1/2}(1-e^{-1/2}) \text{ as } p \downarrow 0$$

we see

$$\liminf_{p \downarrow 0} \ \text{var } g(\mathbf{X}(0)) > 0. \tag{24}$$

So the issue is to upper bound $\mathcal{E}(g,g)$. This is done in the following three lemmas, whose proofs are deferred. In brief, the idea is to define a notion of "good" configurations, and estimate separately the contributions to $\mathcal{E}(g,g)$ from transitions involving good and not-good configurations.

Write $a = \lceil 4 \log 1/p \rceil$. Call a pair (S,i) *admissible* if $i \in S \subseteq \{1,2,\ldots,n\}$ and either $i-1 \in S$ or $i=1$. Call an admissible pair *good* if there exist $k_1, k_2 \in S$ such that
(i) the interval $[k_1,k_2]$ contains i and (if $i \neq 1$) contains $i-1$.
(ii) If $k_2 < 2k_1 - a$ then S does not intersect $[k_1-b,k_1) \cup (k_2,k_2+b]$, where $b = k_2 - k_1 + a < k_1$.
(iii) If $k_2 \geq 2k_1 - a$ then $k_2 \leq \frac{n-a}{2}$ and S does not intersect $(k_2, 2k_2+a]$.

Let $\mathbf{S}_i$ be the random subset of $\{1,2,\ldots,n\}$ containing site i, and (if $i \neq 1$) containing site $i-1$, and where each other site j is in $\mathbf{S}_i$ with probability p, independently as j varies. So $(\mathbf{S}_i, i)$ is admissible, by construction.

Lemma 17

$$P((\mathbf{S}_i, i) \textit{ is not good }) \leq \alpha(p),$$

where $\alpha(p) > 0$ *is a function satisfying*

$$\log \alpha(p) \sim -\frac{\log^2(1/p)}{2\log 2} \ \textit{as } p \downarrow 0.$$

Next consider two CRJ processes started from two different initial configurations. One can couple (i.e. define jointly) the two processes so that, whenever a particle at site i coalesces at some time t' with a particle at site j in one process, if sites i and j are occupied and the intervening sites unoccupied in the other process, then at the same time t' the particle at i coalesces with the particle at j in the other process. The remaining details of the coupling are unimportant.

Lemma 18 *Let (S, i) be admissible and good, and let $|S| \geq 2$. In the coupling of the CRJ processes started from S and from $S \setminus \{i\}$,*

$$P(L(S) \neq L(S \setminus \{i\})) \leq \beta(p)$$

where $\beta(p) > 0$ is a function satisfying

$$\log \beta(p) \sim -2 \log^2(1/p) \text{ as } p \downarrow 0.$$

Lemma 19 $\mathcal{E}(g, g) \leq \alpha(p) + \beta(p)$.

Lemma 19 and the $p \downarrow 0$ asymptotics in Lemmas 17 and 18, combined with the variational characterization (3) and the variance bound (24), establish Theorem 1(b).

5.2 Proofs of the lemmas

Proof of Lemma 19. Consider the restriction to sites $\{1, 2, \ldots, n\}$ of the East process. Write π for the Bernoulli(p) stationary distribution. The possible "upwards" transitions are exactly the transitions $S \setminus \{i\} \to S$ for admissible (S, i). Here we identify a configuration $\mathbf{x} \in \{0, 1\}^n$ with the subset $S_{\mathbf{x}} := \{i : x_i = 1\}$. The stationary flow rate for such a transition is exactly $p\pi(S \setminus \{i\})$ if $i \geq 1$, and at most this quantity if $i = 1$. Since the contributions to $\mathcal{E}(g, g)$ from upwards and downwards transitions are equal,

$$\mathcal{E}(g, g) \leq \sum_S \sum_{i:(S,i) \text{ admissible}} p\pi(S \setminus \{i\}) \ (g(S) - g(S \setminus \{i\}))^2.$$

Recall the definition of $\mathbf{S}_i$ in Lemma 17. Since $P(\mathbf{S}_i = S) = \pi(S)/p^2$ for admissible (S, i), and $\pi(S) = \pi(S \setminus \{i\})p/(1 - p)$, we find

$$\mathcal{E}(g, g) \leq p^2(1 - p) \sum_S \sum_{i:(S,i) \text{ admissible}} P(\mathbf{S}_i = S) \ (g(S) - g(S \setminus \{i\}))^2.$$

Bounding $\sum_i(\cdot)$ by $n \max_i(\cdot)$ and observing $np^2(1 - p) < 1$,

$$\mathcal{E}(g, g) \leq \max_i \ E\left[(g(\mathbf{S}_i) - g(\mathbf{S}_i \setminus \{i\}))^2 \ 1_{((\mathbf{S}_i, i) \text{ admissible})}\right].$$

Recall $0 \leq g \leq 1$. By Lemma 17 the contribution to the expectation from the event where $(\mathbf{S}_i, i)$ is *not* good is at most $\alpha(p)$. By Lemma 18 the contribution to the expectation from the event where $(\mathbf{S}_i, i)$ *is* good is at most $\beta(p)$. So

$$\mathcal{E}(g, g) \leq \alpha(p) + \beta(p)$$

as required. Note that the requirement $|S| \geq 2$ in Lemma 18 eliminates only the case $(S, i) = (\{1\}, 1)$, which makes zero contribution to $\mathcal{E}(g, g)$.

Proof of Lemma 18. Let k_1, k_2 be as in the definition of *good*. Consider first case (ii), where $k_2 < 2k_1 - a$. Define t_0 by

$$t_0 p^b = p^{a/2}; \quad b := k_2 - k_1 + a.$$

Consider the CRJ process started from S. The distance from k_1 to the nearest particle to the left (i.e. at a lower-numbered site) is at least b, so

$$P(\text{ particle at } k_1 \text{ dies before time } t_0) \leq t_0 p^b = p^{a/2}.$$

Similarly, the distance from k_2 to the nearest particle to the right is at least b, so

$$P(\text{ some particle starting to the right coalesces with any}$$
$$\text{particle starting in } [k_1, k_2] \text{ before time } t_0) \quad \leq t_0 p^b = p^{a/2}.$$

Now assume that the particle at site k_1 has not died before time t_0. Then the chance that the particle initially at site k_2 has *not* coalesced with the particle at site k_1 by time t_0 is at most

$$(k_2 - k_1) \exp\left(-t_0 p^{k_2 - k_1}/(k_2 - k_1)\right). \tag{25}$$

To argue (25), divide the time interval $[0, t_0]$ into $k_2 - k_1$ subintervals of length $t_0/(k_2 - k_1)$. In order for the event in question to occur, in one of these subintervals the particle initially at site k_2 must not coalesce with any other particle. But the coalescence rate is at least $p^{k_2 - k_1}$, so the chance of non-coalescence over a subinterval is at most $\exp(-p^{k_2 - k_1} t_0/(k_2 - k_1))$.

Since $k_2 - k_1 < n \leq 1/p$, the quantity in (25) is at most $p^{-1} \exp(-p^{1-a})$. Combining these estimates, the chance that the event

> (*) at time t_0 all the particles initially in $[k_1, k_2]$, and no other particles, have coalesced into a particle currently at site k_1

fails is at most $\beta(p)/2$, where $\beta(p) := 4p^{a/2} + 2p^{-1}\exp(-p^{1-a})$. Now the same argument gives the same bound for the CRJ process started from $S \setminus \{i\}$, replacing k_2 by $i-1$ in the case $k_2 = i$. It follows that, in the coupling, outside an event of probability $\beta(p)$ the two processes are equal at time t_0, implying that $L(S) = L(S \setminus \{i\})$. This establishes Lemma 18 in the case (ii).

Case (iii) is similar. Take $t_0 = p^{k_2+a/2}$ and consider the event

> (**) at time t_0 all the particles initially in $[1, k_2]$, and no other particles, have coalesced into a particle currently at site 0.

Arguing as above, the chance that event (**) fails can be bounded by $p^{a/2} + p^{-1}\exp(-p^{1-a})$. Here the first term bounds the chance that some particle initially to the right of k_2 (and hence, by (iii), to the right of $2k_2+a$) coalesces with the particle initially at k_2; the second term bounds the chance that the particle initially at k_2 has not coalesced with the particle at site 0. The remainder of the argument follows the previous case.

Proof of Lemma 17. We start by examining some deterministic properties of an admissible pair (S, i) which is *not* good. We do the case $i > 1$; the case $i = 1$ is similar. Set $[k_1(0), k_2(0)] = [i-1, i]$ and inductively for $m = 0, 1, 2, 3, \ldots$ specify

(a) if $k_2(m) < 2k_1(m) - a$ then

$$[k_1(m+1), k_2(m+1)] = [j, k_2(m)] \text{ or } [k_1(m), j]$$

where j is the site in $S \setminus [k_1(m), k_2(m)]$ closest to the interval $[k_1(m), k_2(m)]$, breaking ties arbitrarily.

(b) If $2k_1(m) - a \le k_2(m) \le \frac{n-a}{2}$ then

$$[k_1(m+1), k_2(m+1)] = [k_1(m), j]$$

where j is the site in $S \cap (k_2(m), n]$ closest to $k_2(m)$.

The fact that (S, i) is not good implies that no pair $\{k_1(m), k_2(m)\}$ can satisfy conditions (ii,iii), and this implies that the inductive construction makes sense (i.e. the required j's exist at each step) and that $(k_1(m), k_2(m))$ is well-defined for all $0 \le m \le m_1$, where

$$m_1 := \min\{m : k_2(m) \ge 2k_1(m) - a \text{ and } k_2(m) > \tfrac{n-a}{2}\}.$$

Further, for $0 \le m < m_1$:

(c) if $k_2(m) < 2k_1(m) - a$ then $k_2(m+1) - k_1(m+1) \le a + 2(k_2(m) - k_1(m))$
(d) if $k_2(m) \ge 2k_1(m) - a$ then $k_2(m+1) \le 2k_2(m) + a$.

Lemma 20 *Define*

$$\begin{aligned} b(m) &= k_2(m) - k_1(m) \textit{ if } k_2(m) < 2k_1(m) - a \\ &= k_2(m) \textit{ if } k_2(m) \ge 2k_1(m) - a. \end{aligned}$$

Then

$$a + b(m) \le 2^{m+1}(a+1), \ m \le m_1.$$

Moreover

$$m_1 \ge m_0 := \max\{m : \ 2^m(a+1) \le \tfrac{n-a}{2}\}.$$

Proof. By (c,d), the inequality

$$b(m+1) \le a + 2b(m)$$

holds for all $0 \le m < m_1$ *except* perhaps for the first value of m (m^*, say) that $k_2(m^*+1) \ge 2k_1(m^*+1) - a$. For this particular value

$$\begin{aligned} k_2(m^*+1) &= k_1(m^*+1) + (k_2(m^*+1) - k_1(m^*+1)) \\ &\le k_1(m^*+1) + a + 2b(m^*) \\ &\le \tfrac{k_2(m^*+1)+a}{2} + a + 2b(m^*) \end{aligned}$$

and rearranging gives

$$b(m^*+1) = k_2(m^*+1) \le a + 2(a + 2b(m^*)).$$

In other words, if $\bar{b}(m)$ solves the recursion

$$\bar{b}(m+1) = a + 2\bar{b}(m); \quad \bar{b}(0) = 1$$

then inductively

$$\begin{aligned} b(m) &\le \bar{b}(m), \ m \le m^* \\ b(m^*+1) &\le \bar{b}(m^*+2) \\ b(m) &\le \bar{b}(m+1), \ m^* < m \le m_1. \end{aligned}$$

But explicitly

$$\bar{b}(m) = 2^m + a(2^m - 1)$$

and so

$$a + b(m) \le a + \bar{b}(m+1) = 2^{m+1}(1+a)$$

establishing the first inequality in the lemma. For the second inequality, from the definition of m_1 we have $b(m_1) > \frac{n-a}{2}$ and hence $2^{m_1+1}(1+a) > \frac{n-a}{2}$. So $m_1 \ge m_0$ by definition of m_0. ■

Returning to the proof of Lemma 17, consider the random set $\mathbf{S}_i$. In order that $(\mathbf{S}_i, i)$ be not good it is necessary that the random $[k_1(m), k_2(m)]$, $0 \le m \le m_0$ constructed by (a,b) are well-defined and satisfy (c,d). The conditional probability (α_m, say) this holds for $k_1(m+1), k_2(m+1)$ is at most

(case (a,c)): $2(a + k_2(m) - k_1(m))p = 2(a + b(m))p$

(case (b,d)): $(a + k_2(m))p = (a + b(m))p$.

By Lemma 20 $a + b(m) \le 2^{m+1}(1+a)$ and so $\alpha_m \le 2^{m+2}(1+a)p$. So the unconditional probability that $(\mathbf{S}_i, i)$ is not good is at most

$$\begin{aligned} \alpha(p) &:= \prod_{m=0}^{m_0-1} 2^{m+2}(a+1)p \\ &\le 2^{(m_0+1)(m_0+2)/2}((a+1)p)^{m_0}. \end{aligned}$$

From the definition of m_0 we have

$$2^{m_0+1} \le \frac{n}{a+1} \le \frac{1}{p(a+1)}$$

leading to

$$\alpha(p) \le \left(p(a+1)\right)^{m_0/2-1}.$$

Then as $p \downarrow 0$

$$\log \alpha(p) \sim \tfrac{1}{2} m_0 \log p \sim \tfrac{1}{2} \log_2(1/p) \times \log p$$

establishing Lemma 17.

5.3 Remarks on the proof of the lower bound

There is a shorter argument which leads to a cruder lower bound. Take $p = 2^{-m}$ and sites $\{0, 1, 2, \ldots, 2^m\}$. Apply the variational characterization (3) to $g = 1_A$ where A is the set of configurations reachable from the basic (only site 0 occupied) configuration by paths using no more than $m-2$ extra particles. Using straightforward calculations, and combinatorial lemmas analogous to Lemma 2, one can prove that for each $a > 0$

$$\tau(p) \ge \left(\tfrac{1}{p}\right)^{a \log\log(1/p)} \text{ for all sufficiently small } p.$$

Heuristics along these lines were given in [13].

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A Appendix: some technical background

The East process takes values in the space $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$. Give this space the usual product topology and σ-algebra, and let π be product Bernoulli(p) measure. Write L^2 for the space of π-square-integrable functions $f : \mathcal{X} \to R$ and write $\mathcal{C}$ for the space of continuous functions $f : \mathcal{X} \to R$. Finally let

$$\mathcal{D}_1 = \{f \in \mathcal{C} : \sum_k \sup_{\mathbf{x}} |f(\mathbf{x}^k) - f(\mathbf{x})| < \infty\}$$

where $\mathbf{x}^k$ is the configuration obtained from $\mathbf{x}$ by flipping the k'th coordinate. The space $\mathcal{D}_1$ includes the space $\mathcal{D}$ of functions depending on only finitely many coordinates. It will serve as a core (see e.g. Liggett [11] Chapter 1.3) for the generator of the East process.

For $k \in \mathbb{Z}$ and $\mathbf{x} \in \mathcal{X}$ define a measure on $\{0,1\}$ by

$$c_k(\mathbf{x}, y) = (1-p)1_{(y=1, x_{k-1}=1)} + p1_{(y=0, x_{k-1}=1)}.$$

Define a linear operator Ω on $\mathcal{D}$ by

$$\Omega f(\mathbf{x}) = \sum_{k=-\infty}^{\infty} \sum_{y=0}^{1} 1_{(x_k=y)} \left(f(\mathbf{x}^k) - f(\mathbf{x})\right) \ c_k(\mathbf{x}, y).$$

Theorem 3.9 of [11] shows (the conditions (3.3) and (3.8) being easy to check) that the closure $\bar{\Omega}$ of Ω is a generator of a Markov semigroup S_t on $\mathcal{C}$. This extends ([11]) Proposition 4.1) to a generator and semigroup on L^2 with $\mathcal{D}_1$ as core; denote these also by $\bar{\Omega}$ and S_t. The semigroup specifies an $\mathcal{X}$-valued stochastic process $\mathbf{X}_t$ such that

$$E(f(\mathbf{X}_t)|\mathbf{X}_0 = \mathbf{x}) = S_t f(\mathbf{x}); \quad t \in [0, \infty), \ f \in L^2.$$

This is a precise construction of the East process. The operator $\bar{\Omega}$ is a self adjoint unbounded operator on L^2 with $\mathcal{D}_1$ as a core ([11] Chapter 4.4). The spectral theorem implies

$$\bar{\Omega} = \int_0^\infty \lambda G(d\lambda)$$

where $G(\lambda)$ is a resolution of the identity; that is, a family of projections on L^2 satisfying

$$\begin{aligned} G(\lambda_1)G(\lambda_2) &= G(\lambda_1 \wedge \lambda_2) \\ \lim_{\lambda\to\infty} G(\lambda)f &= f \\ \lim_{\lambda\to 0} G(\lambda)f &= G(0)f. \end{aligned}$$

Writing $\sigma(\bar{\Omega})$ for the support of $G(\cdot)$, the spectral gap is defined as

$$\mathrm{gap}(\bar{\Omega}) := \min\{\lambda > 0 : \ \lambda \in \sigma(\bar{\Omega})\}.$$

When we have weak convergence of reversible processes (generators $\bar{\Omega}_n$ and $\bar{\Omega}$ say), we would like to conclude that

$$\mathrm{gap}(\bar{\Omega}) \geq \limsup_n \mathrm{gap}(\bar{\Omega}_n). \tag{26}$$

Reed and Simon [17] show that (26) holds provided the generators $\bar{\Omega}_n, \bar{\Omega}$ have $\mathcal{D}$ as a common core and provided

$$||\bar{\Omega}_n f - \bar{\Omega} f||_2 \to 0 \ \forall f \in \mathcal{D}_1.$$

This can readily be checked for the East process on $\{0, 1, 2, \ldots, n\}$ and the East process on $\{0, 1, 2, \ldots\}$. In the paper we applied (26) to the wave process (at the end of sections 3.1 and 3.3), but this is just a similar argument.