Talk based on paper with Persi Diaconis

The Asymmetric One-Dimensional Constrained Ising Model: Rigorous Results

to appear in *J. Statistical Physics*.

Preprint and copy of these slides on my homepage.

The East Process

$$\_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_$$

Parameter $0 < p < 1$. Equivalent descriptions:

$$\underline{\bullet}\ \_ \quad \to \quad \underline{\bullet}\ \underline{\bullet} \quad \text{rate } p$$
$$\underline{\bullet}\ \underline{\bullet} \quad \to \quad \underline{\bullet}\ \_ \quad \text{rate } 1-p.$$

or: each particle, after exponential(1) random time, sends a "pulse" to site on its East; that site is reset via

$$P(\text{occupied}) = p, \ \ P(\text{unoccupied}) = 1 - p.$$

Process is time-reversible, stationary distribution i.i.d. Bernoulli(p).

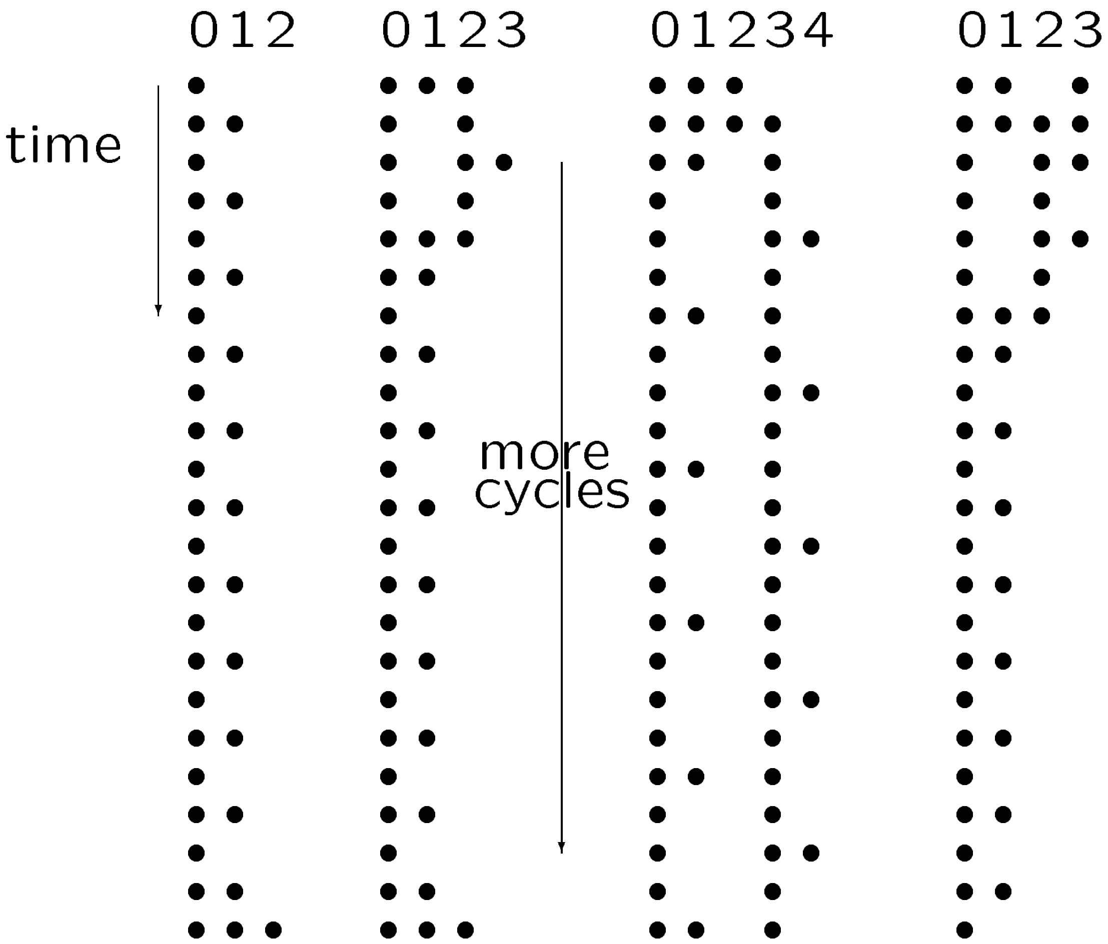
Seek to quantify the heuristic observation

♣ for small p, the East process takes a long time to change substantially.

East process is simple prototype of family of constrained Ising-type processes which have been studied in physical chemistry: "supercooled liquid near the glass transition".

Mathematically, it's a nice example of a simply-describable process which converges slowly to equilibrium.

For small p, particles are typically isolated. Next slide shows (schematically) a realization of the East process from an isolated particle.


0 1 2
0 1 2 3
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time
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cycles

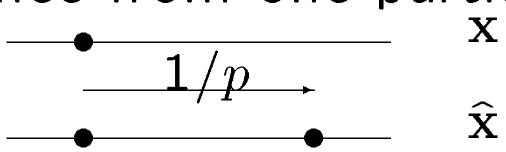
Study relaxation time $\tau(p) = 1/(\text{spectral gap})$ of the East process $(\mathbf{X}(t), 0 \le t < \infty)$:

$$\max_{f,g} \operatorname{cor}(f(\mathbf{X}(0)), g(\mathbf{X}(t))) = \exp(-t/\tau(p)).$$

A 3-line very rough argument suggests

$$\tau(p) \approx \left(\frac{1}{p}\right)^{\log_2 1/p} \text{ as } p \downarrow 0.$$

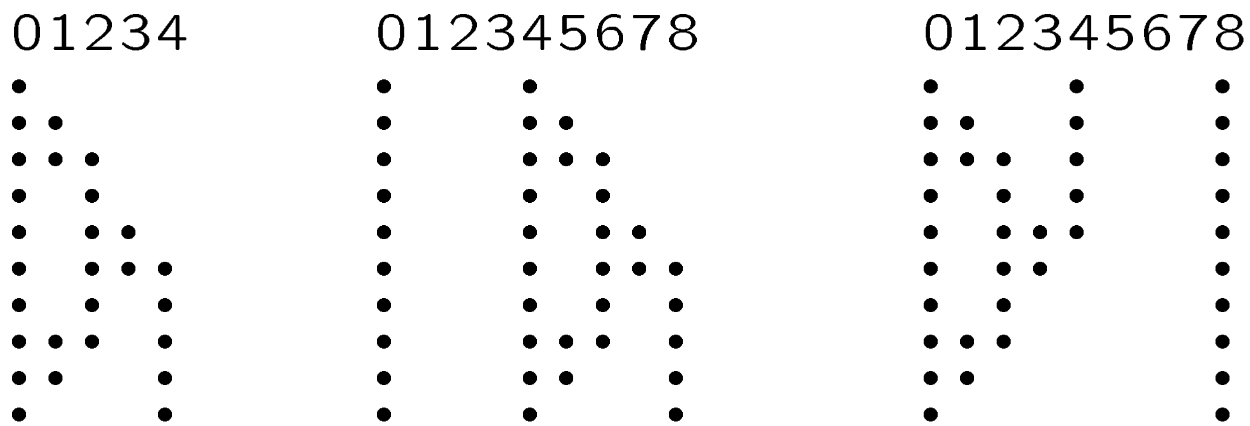
1. $\tau(p) \approx$ time for influence from one particle to reach distance $\approx 1/p$



2. Let $h(m)$ be minimum, over all paths from configuration $\mathbf{x}$ to $\hat{\mathbf{x}}$, of the maximum number of particles in any intermediate configuration. Then (next slide) $h(m) \sim \log_2 m$.

3. Potential barrier between configurations $\mathbf{x}$ and $\hat{\mathbf{x}}$: need to pass through configurations of chance $q = p^{h(1/p)}$. LD heuristics suggest time required $\approx 1/q$.

A path of possible transitions



Rephrase conclusion of heuristic as

$$\log \tau(p) \sim \frac{1}{\log 2}\ \log^2(1/p).$$

Theorem. As $p \downarrow 0$

$$\begin{aligned}\log \tau(p) &\le \frac{1+o(1)}{\log 2}\ \log^2(1/p)\\ &\ge \frac{\frac{1}{2}-o(1)}{\log 2}\ \log^2(1/p).\end{aligned}$$

Proofs are nice mixture of techniques.

Proof of upper bound will use Poincaré comparison with a certain long-range "wave" process – cf. Holley (1985).

1. Analyze relaxation time of the long-range process using coupling and exponential martingales.

2. Make the comparison using minimum-energy paths.

Proof of lower bound uses the variational characterization

$$\tau = \sup_g \frac{\text{var } g}{\mathcal{E}(g,g)}.$$

But (unusual) not easy to guess good g. We end up by

3. defining g implicitly in terms of a certain coalescing random jumps process.

The East process on sites Z^+, site 0 always occupied.

$$\begin{array}{l} \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_ \\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_\ \_ \end{array}$$

Bottom configuration is <u>extension</u> of top configuration. Can couple to preserve that relationship. In particular, can couple

(*) the process $\mathbf{X}^0(t)$ started with only site 0 occupied

(**) the stationary process.

At t the two processes agree on sites 0 through $R(t)$ = rightmost occupied site of $\mathbf{X}^0(t)$. So, restricting to sites $[0, m]$ for fixed m,

$$||P(\mathbf{X}^0(t) \in \cdot) - \pi(\cdot)|| \leq P(R(t) < m)$$

where π is (restricted) stationary dist. By the elementary relationship between asymptotic variation distance and spectral gap,

$R(t)$ for process $\mathbf{X}^0$

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(*) If $P(R(t) < m) = O(e^{-\lambda t})$ as $t \to \infty$; $\forall m$ then spectral gap $\geq \lambda$.

But can't analyze $R(t)$ for East process; so invent long-range "wave" process for which (*) holds and $R(t)$ can be analyzed.

The wave process. Each particle, after exponential(1) random time, sends a "wave" to cover the $10/p$ sites on its East; those sites are reset to i.i.d. Bernoulli (p).

Can prove the wave process has spectral gap $\geq \lambda = 3/10$ by showing

$$\exp(\lambda t - \theta R(t)) \text{ is a supermartingale}$$

for some $\theta > 0$.

Set $2^m \approx 1/p$. Consider typical transition of wave process.

sites 0 $1/p$

$\mathbf{x}$ • _ _ _ _ _ _ • _ _ _ _ _ _ _ _ _ _ • _ _ _ _ _ _

$\mathbf{y}$ • _ _ _ _ _ _ _ _ _ _ • _ _ _ _ _ _ _ _ _ _ _ _ _

Can expand transition as path of transitions of East process, with path-length $2 \cdot 3^m$ and using configurations with at most $\max(\mathbf{x}, \mathbf{y}) + m + 1$ particles. Comparison argument of Diaconis & Saloff-Coste (1993) gives

$$\frac{\tau(\text{East})}{\tau(\text{Wave})} \leq 2 \cdot 3^m \times \max_{\mathbf{x}, \hat{\mathbf{x}}} \frac{\text{induced flow } \mathbf{x} \to \hat{\mathbf{x}}}{\text{East flow } \mathbf{x} \to \hat{\mathbf{x}}}$$

$$\leq \text{poly}(1/p) \times (1/p)^m \approx (1/p)^{\log_2 m}.$$

The Lower Bound

Need to choose g and apply

$$\tau = \sup_g \frac{\text{var } g}{\mathcal{E}(g,g)}$$

$$\mathcal{E}(g,g) := \tfrac{1}{2} \lim_{t \downarrow 0} E(g(\mathbf{X}(t)) - g(\mathbf{X}(0)))^2.$$

Idea: occupied site with large gap to left persists for long time.

But hard to convert to definition of g. We use indirect approach. Take sites 0 thru $1/p$, with site 0 always occupied. From some initial configuration $\mathbf{x}$ define a coalescing process by: each particle (site j say) merges with nearest particle to its left (site i say) at rate p^{j-i}. Ultimately only one particle away from site 0. Define

$g(\mathbf{x}) = P(\text{final particle in left half of site-interval}).$

$g(\mathbf{x}) = P$(final particle in left half of site-interval).

Immediate: var $g(\cdot) > \delta > 0 \ \forall p$.
So want to show $\mathcal{E}(g,g)$ is very small, of order $p^{\log(1/p)}$. Consider typical transition of East process:

$$\begin{array}{ll} \mathbf{x} & \_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_ \\ \hat{\mathbf{x}} & \_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{\bullet}\ \underline{\bullet}\ \_\ \_\ \_\ \underline{\bullet}\ \_\ \_\ \_\ \_\ \_\ \_ \end{array}$$

$|g(\hat{\mathbf{x}}) - g(\mathbf{x})| \leq$ chance first move of coalescing process started at $\hat{\mathbf{x}}$ is <u>not</u> $\bullet\bullet \ \rightarrow \ \bullet\circ$.

For typical $\mathbf{x}$ this chance is $p^{1/p}$; sufficiently small. But we need bound for all configurations; technically hard.

Final Remarks

1. Everyone knows that one-dimensional Ising-type models have non-zero spectral gap. But standard theory assumes non-zero flip rates; in fact not previously known that $\tau(p) < \infty$.

2. East model is very special case of very general constructions. Take

- any reversible spin system $\mathbf{X}$
- any family of neighborhoods $\mathcal{N}_i$ of sites i
- any subset $S_i \subseteq \{-1, +1\}^{\mathcal{N}_i \setminus \{i\}}$ of spin configurations in the neighborhood excluding i itself.

Define the *constrained* process by: flip rate at site i is

same as for $\mathbf{X}$ if n'hood configuration $\in S_i$;
$= 0$ if not.