The Percolation Process on a Tree Where Infinite Clusters are Frozen

David J. Aldous*
Department of Statistics
University of California
367 Evans Hall # 3860
Berkeley CA 94720-3860
aldous@stat.berkeley.edu
http://www.stat.berkeley.edu/users/aldous

January 20, 1999

Abstract

Modify the usual percolation process on the infinite binary tree by forbidding infinite clusters to grow further. The ultimate configuration will consist of both infinite and finite clusters. We give a rigorous construction of a version of this process and show that one can do explicit calculations of various quantities, for instance the law of the time (if any) that the cluster containing a fixed edge becomes infinite. Surprisingly, the distribution of the shape of a cluster which becomes infinite at time $t > 1/2$ does not depend on t; it is always distributed as the incipient infinite percolation cluster on the tree. Similarly, a typical finite cluster at each time $t > 1/2$ has the distribution of a critical percolation cluster. This elaborates an observation of Stockmayer (1942).

AMS 1991 subject classification: 60K35, 05C80

*Research supported by N.S.F. Grant DMS96-22859

1 Introduction

Let $\mathbf{T} = (\mathcal{V}, \mathcal{E})$ be the infinite binary tree, wherein each vertex has degree three: $\mathcal{V}$ is the vertex-set and $\mathcal{E}$ is the set of undirected edges. Let $(U_e, e \in \mathcal{E})$ be independent r.v.'s with $U(0,1)$ (uniform on the interval $(0,1)$) law. Setting $\mathcal{B}_t = \{e : U_e \leq t\}$ gives the *percolation process* $(\mathcal{B}_t, 0 \leq t \leq 1)$ on $\mathbf{T}$. This has often been studied (e.g. Grimmett [7] sec. 8.1) as a simple proxy for the more complicated percolation process on $\mathbb{Z}^d$. It is elementary that the clusters (connected edge components) of $\mathcal{B}_t$ can be described in terms of Galton-Watson branching processes, and that infinite clusters exist for $t > 1/2$ but not for $t \leq 1/2$.

The evolution of the percolation process may be described by:

> for each $e \in \mathcal{E}$, at time $t = U_e$ set $\mathcal{B}_t = \mathcal{B}_{t-} \cup \{e\}$.

The purpose of this paper is to study an analogous process of random subsets $\mathcal{A}_t \subseteq \mathcal{E}$ whose evolution $(\mathcal{A}_t, 0 \leq t \leq 1)$ is described informally by: $\mathcal{A}_0$ is empty,

> (*) for each $e \in \mathcal{E}$, at time $t = U_e$ set $\mathcal{A}_t = \mathcal{A}_{t-} \cup \{e\}$ if each end-vertex of e is in a finite cluster of $\mathcal{A}_{t-}$; otherwise set $\mathcal{A}_t = \mathcal{A}_{t-}$.

(A cluster is formally a set of edges, but we use the same word to denote the induced set of vertices). This process is apparently novel, and seems natural enough to warrant study. Our specific motivation is described in section 1.1, and further related work and open problems are discussed in section 5.

Any process satisfying (*) must have $\mathcal{A}_t = \mathcal{B}_t$ for $t \leq 1/2$ but $\mathcal{A}_t \subset \mathcal{B}_t$ for $t > 1/2$. Qualitatively, in the process $\mathcal{A}_t$ the clusters may grow to infinite size but, at the instant of becoming infinite, they are "frozen" in the sense that no extra edges may be connected to an infinite cluster. The final state $\mathcal{A}_1$ will be a random forest on $\mathbf{T}$ with both finite and infinite clusters, such that no two finite clusters are separated by a single edge.

Rigorously speaking, it is not clear that (*) does specify a unique process. In section 3 we give a rigorous construction, summarized as follows.

Theorem 1 *There exists a joint law for $(\mathcal{A}_t, 0 \leq t \leq 1)$ and $(U_e, e \in \mathcal{E})$ such that (*) holds and the joint law is invariant under automorphisms of* $\mathbf{T}$.

Call this $(\mathcal{A}_t)$ the *frozen percolation* process. We conjecture this is the *unique* process satisfying (*), but it seems hard to exclude the possibility that there might exist *non* automorphism-invariant processes satisfying (*). Here are some explicit properties of the frozen percolation process.

Proposition 2 *For a prescribed edge e and vertex v:*
(a) $P($cluster containing e becomes infinite in $[t, t+dt]) = \frac{1}{4t^4}dt,\ \frac{1}{2} \leq t \leq 1$.
(b) $P($cluster containing v becomes infinite in $[t, t+dt]) = \frac{3}{8t^4}dt,\ \frac{1}{2} \leq t \leq 1$.

$$(c) \qquad \begin{aligned} P(e \textit{ in some infinite cluster of } \mathcal{A}_1) &= 7/12 \\ P(e \textit{ in some finite cluster of } \mathcal{A}_1) &= 1/16 \\ P(e \notin \mathcal{A}_1) &= 17/48 \end{aligned}$$

$$(d) \qquad \begin{aligned} P(v \textit{ in some infinite cluster of } \mathcal{A}_1) &= 7/8 \\ P(v \textit{ in some finite cluster of } \mathcal{A}_1) &= 7/64 \\ P(v \textit{ in no cluster of } \mathcal{A}_1) &= 1/64. \end{aligned}$$

The pattern of our argument is as follows. A key distributional recursion is isolated in section 2.1. In section 2.2 we assume the frozen percolation process exists with natural independence and symmetry properties, and derive the formulas in Proposition 2. The calculations use the idea that for a directed edge $\vec{e}$ there may be a first time t that $\mathcal{A}_t$ contains an infinite directed path from $\vec{e}$: call this time $Y_{\vec{e}}$. The random variables $Y_{\vec{e}}$ satisfy the key recursion as $\vec{e}$ varies, from which their law may be determined. In the rigorous construction (section 3) we reverse the argument: first create by fiat random variables $Y_{\vec{e}}$ satisfying the recursion, then use them to define the frozen percolation process. The point of this seemingly illogical order of presentation is that the rigorous argument would appear very mysterious without having seen the results of the heuristics.

In section 4 we study the shape of clusters of the frozen percolation process for $t > 1/2$. Fix an edge $\tilde{e}$. Conditional on $\tilde{e}$ being in a finite cluster of $\mathcal{A}_t$, the cluster has the law of a critical percolation cluster on $\mathbf{T}$ (Proposition 11). Conditional on $\tilde{e}$ being in an infinite cluster of $\mathcal{A}_1$, the cluster has the law of the incipient infinite percolation cluster on $\mathbf{T}$, studied by Kesten [10], and moreover is independent of the time at which the cluster becomes infinite (Theorem 14). This simple structure seems remarkable – we do not have a simple explanation.

We remark that the alternative process $(\mathcal{D}_t)$ defined by

> for each $e \in \mathbf{T}$, at time $t = U_e$ set $\mathcal{D}_t = \mathcal{D}_{t-} \cup \{e\}$ if <u>at least one</u> end-vertex of e is in a finite cluster of $\mathcal{D}_{t-}$; otherwise set $\mathcal{D}_t = \mathcal{D}_{t-}$

is conceptually simpler, because there is a direct criterion in terms of (U_e) for whether a particular edge e enters the process at time $t = U_e$: is either end-vertex of e in a finite cluster of $\{e : U_e \leq t\}$? Häggström [9] discusses $\mathcal{D}_1$ as the *minimal essential spanning forest* on $\mathbf{T}$. In $\mathcal{D}_1$ all clusters are infinite, but ([9] Theorem 4.5) the cluster containing a specified edge does not have the law of the incipient infinite percolation cluster. The frozen percolation process is harder to study because there is apparently no simple criterion, in terms of the U's only, for whether e enters the process at time U_e. In fact it is not obvious from our construction that $\mathcal{A}_1$ is measurable with respect to $(U_e, e \in \mathcal{E})$: see section 5.7.

1.1 Classical polymerization models

Regard a cluster in $\mathbf{T}$ as a polymer made up of "molecular units" capable of forming three bonds. Then the ordinary percolation process $\mathcal{B}_t$ can be regarded as a process of polymerization, where before the critical time there are only finite polymers (the *sol*) and later there are infinite polymers (the *gel*) also. Such models go back to Flory [6], but are usually presented in a different way. Without explicitly mentioning the infinite tree, one gets the same model by envisaging polymers in three-dimensional space and assuming that any possible bond can form regardless of geometric position; without any explicitly specified underlying stochastic process, one can write down and solve equations for $c_i(t)$ = proportion of size-i clusters at time t. See van Dongen [13] for a recent review. In Flory's model, like ordinary percolation, there is interaction between sol and gel, while the variation in which bonds form only between finite polymers was studied by Stockmayer [12], which contains the following passage (without further elaboration).

> As the reaction continues beyond the gel point, the number of small molecules ... decreases, but ... their average size may be shown by substitution into Eq. (15) to retain the constant value ...

Our frozen percolation process is intended as a interpretation of Stockmayer's idea within a precise stochastic model, with Proposition 11 as a formalization of the passage above. Note that Proposition 11 deals with the distribution of the cluster containing a given edge, i.e. the size-biasing of the distribution of a typical cluster.

2 Computational arguments

2.1 The distributional recursion

Set $I = [1/2, 1] \cup \{\infty\}$. Define $\Phi : I \times [0,1] \to I$ by:

$$\begin{aligned} \Phi(x,u) &= x \text{ if } x > u \\ &= \infty \text{ if } x \leq u. \end{aligned}$$

Define a probability law ν on I by

$$\nu(dy) = \tfrac{1}{2y^2}\ dy,\ \tfrac{1}{2} \leq y \leq 1, \quad \nu(\infty) = \tfrac{1}{2} \tag{1}$$

or equivalently by

$$\nu(y,\infty] = \tfrac{1}{2y},\ \tfrac{1}{2} \leq y \leq 1.$$

Consider the following property for a probability law μ on I.

If (Y_1, Y_2, U) are independent, each Y_i having law μ and U having $U(0,1)$ law,

$$\text{then } \Phi(\min(Y_1,Y_2),U) \text{ has law } \mu. \tag{2}$$

Lemma 3 *Let μ be a probability law on I which is non-atomic on $[1/2,1]$. Then μ has property (2) if and only if for some $1/2 \leq x_0 \leq 1$,*

$$\mu(dx) = \tfrac{1}{2x^2}\ dx,\ \ \tfrac{1}{2} < x \leq x_0;\ \ \mu(\infty) = \frac{1}{2x_0}.$$

So in particular the law ν at (1) has property (2).
Proof. From the definition of Φ, a law μ on I has property (2) iff the distribution function F of μ satisfies

$$F(x) = P(U < \min(Y_1,Y_2) \leq x),\ \ \tfrac{1}{2} \leq x \leq 1.$$

In the non-atomic case this is equivalent to

$$dF(x) = 2x(1-F(x))\ dF(x),\ \ \tfrac{1}{2} \leq x \leq 1$$

and hence equivalent to

$$F(x) = 1 - \tfrac{1}{2x} \text{ on } [1/2,1] \cap \text{support}(\mu).$$

Because the function $1 - \frac{1}{2x}$ is strictly increasing, this can only happen if the support is of the form $[\frac{1}{2}, x_0]$ for some x_0. Reversing the argument, such laws do indeed have property (2). $\square$

2.2 The heuristic argument

In this section we assume the frozen percolation process exists and has the natural invariance and independence properties, and we proceed to do calculations. The results of some of these calculations motivate the rigorous construction in section 3.

It is convenient to start by studying a modified tree $\widetilde{\mathbf{T}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$, wherein one distinguished vertex (the *root*) has degree 1, and the other vertices have degree 3. This $\widetilde{\mathbf{T}}$ is sometimes called the *planted* binary tree. Write $\tilde{e} = (\text{root}, v_*)$ for the edge at the root. Clearly $\widetilde{\mathbf{T}}$ is isomorphic to certain subtrees of $\mathbf{T}$, in the following way. Distinguish an edge in $\mathcal{E}$ and call it $\tilde{e} = (\text{root}, v_*)$, then delete the other two edges at "root" and their induced subtrees: the resulting subtree is isomorphic to $\widetilde{\mathbf{T}}$.

As in section 1, let $(U_e, e \in \widetilde{\mathcal{E}})$ be independent r.v.'s distributed uniformly on $[0,1]$, and consider a frozen percolation process on $\widetilde{\mathbf{T}}$. Let Y be the time at which the component containing $\tilde{e}$ becomes infinite, with $Y = \infty$ if never. Write e_1 and e_2 for the other two edges at v_*, write $\widetilde{\mathbf{T}}_1$ for the subtree containing e_1 obtained by deleting $\tilde{e}$ and e_2 and their induced subtrees. Let Y_1 be the time at which, in frozen percolation on $\widetilde{\mathbf{T}}_1$, the cluster containing e_1 becomes infinite. Define $\widetilde{\mathbf{T}}_2$ and Y_2 similarly. Then in frozen percolation on $\widetilde{\mathbf{T}}$, the cluster containing vertex v_* becomes infinite at time $\min(Y_1, Y_2)$. At that time, if edge $\tilde{e}$ has already appeared, i.e. if $U_{\tilde{e}} < \min(Y_1, Y_2)$, then edge $\tilde{e}$ joins an infinite component; otherwise $\tilde{e}$ never enters the process. Thus

$$Y = \Phi(\min(Y_1, Y_2), U_{\tilde{e}}). \tag{3}$$

Since the trees $\widetilde{\mathbf{T}}_i$ are isomorphic to $\widetilde{\mathbf{T}}$ we expect Y_1 and Y_2 to have the same law as Y. So this law satisfies (2), and Lemma 3 strongly suggests it is the law (1). Assuming this is true, we proceed to do the calculations leading to Proposition 2. Consider the frozen percolation process on $\mathbf{T}$. Fix an edge e and let Z be the time at which e enters an infinite cluster ($Z = \infty$ if never). Write e_1, e_2, e_3, e_4 for the edges adjacent to e, $\widetilde{\mathbf{T}}_1, \ldots, \widetilde{\mathbf{T}}_4$ for the corresponding subtrees isomorphic to $\widetilde{\mathbf{T}}$, and $(Y_i, 1 \le i \le 4)$ for the times at which e_i enters an infinite cluster of the frozen percolation process restricted to $\widetilde{\mathbf{T}}_i$. Then

$$\begin{aligned} Z &= \min(Y_1, Y_2, Y_3, Y_4) \text{ if } U_e < \min(Y_1, Y_2, Y_3, Y_4) \\ &= \infty \text{ if not.} \end{aligned} \tag{4}$$

We can therefore obtain the density f_Z of Z on $\frac{1}{2} \le x \le 1$ in terms of the law (1) of Y:

$$f_Z(x) = x \times 4 \tfrac{d\nu}{dx} \nu^3(x, \infty) = 4x \ \tfrac{1}{2x^2} (\tfrac{1}{2x})^3 = \tfrac{1}{4x^4}. \tag{5}$$

This is assertion (a) of Proposition 2. So

$$P(e \text{ in some infinite cluster of } \mathcal{A}_1) = P(Z \leq 1) = \int_{1/2}^{1} \tfrac{1}{4x^4} dx = \tfrac{7}{12}.$$

Next, note that the event $\{e$ in some finite cluster of $\mathcal{A}_1\}$ is the event $\{Y_i = \infty, 1 \leq i \leq 4\}$. Clearly the latter event has chance $\nu^4(\infty) = (1/2)^4 = 1/16$. These calculations establish assertion (c) of Proposition 2, since the three events in question are exclusive and exhaustive.

Now fix a vertex v. Let $\widetilde{\mathbf{T}}_1, \widetilde{\mathbf{T}}_2, \widetilde{\mathbf{T}}_3$ be the subtrees induced by the three edges e_1, e_2, e_3 at v, and $(Y_i, 1 \leq i \leq 3)$ the times at which, in frozen percolation on $\widetilde{\mathbf{T}}_i$, the cluster containing e_i becomes infinite. Then in the frozen percolation process on $\mathbf{T}$,

$$\text{the time at which } v \text{ joins an infinite cluster is distributed as } \min(Y_1, Y_2, Y_3). \tag{6}$$

By (1) the density of this minimum is

$$3 \tfrac{d\nu}{dy}\ \nu^2(y, \infty) = 3 \tfrac{1}{2y^2} (\tfrac{1}{2y})^2 = \tfrac{3}{8y^4},$$

establishing assertion (b) of Proposition 2. Moreover

$$P(v \text{ in some infinite cluster of } \mathcal{A}_1) = 1 - P^3(Y = \infty) = 1 - (1/2)^3 = 7/8.$$

Similarly

$$P(v \text{ in no cluster of } \mathcal{A}_1) = q^3$$

where q is the chance that, in frozen percolation on $\widetilde{\mathbf{T}}$, the special edge $\tilde{e}$ is not in $\mathcal{A}_1$. Now in the notation of (3)

$$q = P(U_{\tilde{e}} \geq \min(Y_1, Y_2)) = \int_{1/2}^{1} P(\min(Y_1, Y_2) \leq x) dx = \int_{1/2}^{1} (1 - (\tfrac{1}{2x})^2) dx = \tfrac{1}{4}.$$

Thus $P(v$ in no cluster of $\mathcal{A}_1) = (\frac{1}{4})^3 = \frac{1}{64}$, establishing assertion (d) of Proposition 2.

3 The construction

Recall $\mathbf{T} = (\mathcal{V}, \mathcal{E})$ is the infinite binary tree, and $(U_e, e \in \mathcal{E})$ are independent r.v.'s with $U(0,1)$ law. Each undirected edge e can be identified with two directed edges $\overrightarrow{e}, \overleftarrow{e}$: write $\overrightarrow{\mathcal{E}}$ for the set of directed edges. For $\overrightarrow{e} \in \overrightarrow{\mathcal{E}}$ define $U_{\overrightarrow{e}} = U_{\overleftarrow{e}} = U_e$ where e is the corresponding undirected edge. According to context, (v, w) may denote either the undirected edge or the directed edge. For directed edges we have a natural language of family relationships: the edge $\overrightarrow{e} = (v, w)$ has two *children* of the form (w, x_1) and (w, x_2).

Recall ν is the law (1). Note that direction matters in the following lemma: $Y_{\overrightarrow{e}}$ is typically different from $Y_{\overleftarrow{e}}$.

Lemma 4 *There exists a joint law for $((U_{\overrightarrow{e}}, Y_{\overrightarrow{e}}) : \overrightarrow{e} \in \overrightarrow{\mathcal{E}})$ which is invariant under automorphisms of* $\mathbf{T}$*, and such that for each $\overrightarrow{e} \in \overrightarrow{\mathcal{E}}$*

$$\begin{aligned} Y_{\overrightarrow{e}} & \quad \textit{has law } \nu \\ Y_{\overrightarrow{e}} & = \Phi(\min(Y_{\overrightarrow{e_1}}, Y_{\overrightarrow{e_2}}), U_{\overrightarrow{e}}) \ \textit{a.s.} \end{aligned} \tag{7}$$

where $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are the children of $\overrightarrow{e}$.

Proof. Fix an undirected edge e_0. Fix $h \geq 1$. Let $\overrightarrow{\mathcal{E}}_{\leq h}$ be the set of directed edges whose distance from e_0 is at most h. Let $\overrightarrow{\mathcal{E}}_h$ be the set of directed edges whose distance from e_0 is exactly h and which are directed away from e_0. Take $(Y_{\overrightarrow{e}} : \overrightarrow{e} \in \overrightarrow{\mathcal{E}}_h)$ to be independent of $(U_{\overrightarrow{e}}, \overrightarrow{e} \in \overrightarrow{\mathcal{E}})$ with each $Y_{\overrightarrow{e}}$ having law ν. Use (7) recursively to define $Y_{\overrightarrow{e}}$ for $\overrightarrow{e} \in \overrightarrow{\mathcal{E}}_{\leq h}$. Lemma 3 ensures that each $Y_{\overrightarrow{e}}$ has law ν. As h increases, these joint laws are consistent (again, by Lemma 3) and so the Kolmogorov consistency theorem establishes existence of a joint law for $((U_{\overrightarrow{e}}, Y_{\overrightarrow{e}}) : \overrightarrow{e} \in \overrightarrow{\mathcal{E}})$. Checking invariance is straightforward. □

It is easy to check the following independence properties of the construction above. For a directed edge $\overrightarrow{e}$, consider the set consisting of $\overrightarrow{e}$ and all its descendants; then write $D(\overrightarrow{e})$ for this set of edges, considered as *undirected* edges.

Corollary 5 *For each $i \geq 1$ let $f_{i,1}, f_{i,2}, \ldots \in \mathcal{E}$ and let $\overrightarrow{e_{i,1}}, \overrightarrow{e_{i,2}}, \ldots \in \overrightarrow{\mathcal{E}}$. Write $D_i = \cup_j D(\overrightarrow{e_{i,j}}) \cup \{f_{i,1}, f_{i,2}, \ldots\}$. If the sets D_i are disjoint as i varies then the σ-fields $\sigma(Y_{\overrightarrow{e_{i,j}}}, U_{f_{i,j}}, j \geq 1)$ are independent as i varies.*

By modifying on a null set, we may suppose that in (7) the equality holds always (instead of a.s.) and that the values of $U_e, Y_{\overrightarrow{e_1}}, Y_{\overrightarrow{e_2}}$ (where finite) are surely distinct, and not equal to 1. Suppose $Y_{\overrightarrow{e}} \le 1$. Then exactly one child of $\overrightarrow{e}$ has Y-value $= Y_{\overrightarrow{e}}$. So arguing inductively, associated with $\overrightarrow{e}$ is an infinite ray $\overrightarrow{e} = \overrightarrow{e_0}, \overrightarrow{e_1}, \overrightarrow{e_2}, \ldots$ such that $Y_{\overrightarrow{e_i}} = Y_{\overrightarrow{e}}$ for all i.

We can now state the construction of the frozen percolation process. Essentially, we take the heuristically obvious property (4) as a definition. For an undirected edge e, write $\partial(\{e\})$ for the set consisting of the four edges adjacent to e, each directed away from e. Define

$$\mathcal{A}_1 = \{e \in \mathcal{E} : U_e < \min(Y_{e'} : e' \in \partial(\{e\})\ )\ \}.$$

Then define

$$\mathcal{A}_t = \{e \in \mathcal{A}_1 : U_e \le t\}, \qquad 0 \le t < 1. \tag{8}$$

It is clear that $(\mathcal{A}_t)$ inherits from $(Y_{\overrightarrow{e}})$ the automorphism-invariance property. It is also clear that the only possible time at which e can join the process $(\mathcal{A}_t, 0 \le t \le 1)$ is at time U_e. To complete the proof of Theorem 1 we need a further result. Say a vertex v *percolates* at time t if it is in an infinite cluster of $\mathcal{A}_t$, that is if there exists an infinite ray $v = v_0, v_1, v_2, \ldots$ such that each (v_i, v_{i+1}) is in $\mathcal{A}_t$.

Proposition 6 *Let $t < 1$. A vertex v percolates at time t iff $t \ge \min(Y_{\overrightarrow{e_1}}, Y_{\overrightarrow{e_2}}, Y_{\overrightarrow{e_3}})$, where $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ are the edges at v, directed away from v.*

Granted this result, fix an undirected edge e and consider time $t = U_e < 1$. Let $\overrightarrow{e} = (v, w)$ be a directing of e. By (7), $Y_{\overrightarrow{e}} > t$. By Proposition 6, v is in a finite cluster at time t iff $t < \min(Y_{\overrightarrow{e_2}}, Y_{\overrightarrow{e_3}})$, where $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$ are the other edges at v, directed away from v. Applying the same argument to w, we see that the property

both end-vertices of e are in finite clusters at time t

is equivalent to the property

$$t < \min(Y_{e'} : e' \in \partial(\{e\}).$$

Thus the defining criterion (8) for e joining the frozen percolation process $(\mathcal{A}_t)$ is exactly the same as rule (*). This establishes Theorem 1.

The proof of Proposition 6 requires a series of lemmas.

Lemma 7 *Let $\vec{e_2}$ be a child of $\vec{e_1}$. If e_1 (the undirected edge corresponding to $\vec{e_1}$) has $e_1 \in \mathcal{A}_1$ then $Y_{\vec{e_1}} \leq Y_{\vec{e_2}}$.*

Proof. Let $\vec{e_3}$ be the other child of $\vec{e_1}$. By the recursion (7), the only way it can happen that $Y_{\vec{e_1}} > Y_{\vec{e_2}}$ is if $\min(Y_{\vec{e_2}}, Y_{\vec{e_3}}) \leq U_{\vec{e_1}}$, but in that case $e_1 \notin \mathcal{A}_1$. □

Lemma 8 *If $t < Y_{\vec{e}}$ and $t < 1$ then there is no infinite ray of edges in $\mathcal{A}_t$ starting with $\vec{e}$.*

Proof. Fix $t < t_2 \leq 1$ and a finite path $\vec{e} = \vec{e_0}, \vec{e_1}, \vec{e_2}, \ldots, \vec{e_h}$. Consider the event

$$D = \{Y_{\vec{e}} \geq t_2 \;\text{ and }\; e_i \in \mathcal{A}_t \text{ for } 0 \leq i \leq h\}.$$

On D, for $1 \leq i \leq h$ the fact $e_{i-1} \in \mathcal{A}_1$ implies $U_{e_{i-1}} < \min(Y_{\vec{e_i}}, Y_{\vec{f_i}})$, where $\vec{f_i}$ is the other child of $\vec{e}_{i-1}$, and therefore

$$Y_{\vec{e}_{i-1}} = \min(Y_{\vec{e_i}}, Y_{\vec{f_i}}).$$

On the other hand, on D we have $Y_{\vec{e}} \geq t_2$ and so by applying Lemma 7 repeatedly we see $Y_{\vec{e}_{i-1}} \geq t_2$. So the equality above shows $Y_{\vec{f_i}} \geq t_2$. So

$$P(D) \leq P\left(U_{e_i} \leq t, Y_{\vec{f_i}} \geq t_2,\; 1 \leq i \leq h\right).$$

By Corollary 5 the random variables involved in the right side are independent, and using (1) we find

$$P(D) \leq (t/2t_2)^h.$$

Summing over all 2^h possible paths of length h,

$$P(Y_{\vec{e}} \geq t_2, \exists\ h\text{-edge path in } \mathcal{A}_t \text{ starting with } \vec{e}) \leq (t/t_2)^h. \tag{9}$$

Letting $h \to \infty$,

$$P(Y_{\vec{e}} \geq t_2,\ \exists \text{ infinite path in } \mathcal{A}_t \text{ starting with } \vec{e}) = 0.$$

Letting $t_2 \downarrow t$ establishes the lemma. □

Recall the statement of Proposition 6 and change the notation slightly: write $\vec{e}_{[1]}, \vec{e}_{[2]}, \vec{e}_{[3]}$ for the edges at a vertex v, directed away from v. Lemma 8 establishes half of Proposition 6: if $t < \min(Y_{\vec{e}_{[1]}}, Y_{\vec{e}_{[2]}}, Y_{\vec{e}_{[3]}})$ and $t < 1$ then v does not percolate at time t. The next lemma establishes the other half.

Lemma 9 *If* $t = Y_{\overrightarrow{e}_{[1]}} < \min(Y_{\overrightarrow{e}_{[2]}}, Y_{\overrightarrow{e}_{[3]}})$ *then* v *percolates at time* t *via edge* $\overrightarrow{e}_{[1]}$.

Proof. We may take $t \leq 1$. Since $Y_{\overrightarrow{e}_{[1]}} \leq 1$, using (7) repeatedly there exists an associated infinite ray $\overrightarrow{e}_{[1]} = \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3, \ldots$ such that

$$U_{\overrightarrow{e}_i} < \quad t = Y_{\overrightarrow{e}_i} \qquad\qquad i \geq 1 \tag{10}$$

$$t = Y_{\overrightarrow{e}_i} \quad < Y_{\overrightarrow{f}_i}, \quad i \geq 2 \tag{11}$$

where $\overrightarrow{f}_i$ is the other child of $\overrightarrow{e}_{i-1}$. We need to show $\overrightarrow{e}_j \in \mathcal{A}_t$ for each $j \geq 1$. For $j \geq 2$ this means we need to show

$$U_{\overrightarrow{e}_j} < \min\left(Y_{\overrightarrow{e}_{j+1}}, Y_{\overrightarrow{f}_{j+1}}, Y_{\overrightarrow{f}_j}, Y_{\overleftarrow{e}_{j-1}}\right).$$

The first three terms on the right satisfy the inequality by (10,11), so it is enough to show

$$t \leq Y_{\overleftarrow{e}_{j-1}}, \quad j \geq 2.$$

If this fails, it fails for some minimal j, and $Y_{\overleftarrow{e}_{j-1}} = t' < t$, say. If $j \geq 3$ then by (7) either $Y_{\overrightarrow{f}_{j-1}} = t'$ or $Y_{\overleftarrow{e}_{j-2}} = t'$; but the former is forbidden by (11) and the latter by minimality. If it fails for $j = 2$ then (7) implies either $Y_{\overrightarrow{e}_{[2]}} = t'$ or $Y_{\overrightarrow{e}_{[3]}} = t'$, but this is forbidden by hypothesis. So $\overrightarrow{e}_j \in \mathcal{A}_t$ for all $j \geq 2$. Similarly, to show $\overrightarrow{e}_{[1]} \in \mathcal{A}_t$ it is enough to show

$$t \leq \min\left(Y_{\overrightarrow{e}_2}, Y_{\overrightarrow{f}_2}, Y_{\overrightarrow{e}_{[2]}}, Y_{\overrightarrow{e}_{[3]}}\right)$$

and again this follows from (11) and hypothesis. □

3.1 Complements to the construction

Reconsider the calculations in section 2.2 which led to the formulas stated as Proposition 2. The calculations were based on (3,4,6) and independence properties; these were rigorously established in (7,8), Proposition 6 and Corollary 5. So Proposition 2 is rigorously established.

In the proof of Proposition 6 we made use (at (9)) of the fact $t \neq 1$. A further argument, Lemma 13 below, extends Proposition 6 to $t = 1$, which then implies the corresponding result for edges, which we state as

Corollary 10 *For $e \in \mathcal{E}$ define*

$$Z_e = \min(Y_{e'} : e' \in \partial(\{e\})\ ).$$

Then either $Z_e = \infty$ and $e \notin \mathcal{A}_1$, or e enters the frozen percolation process at time U_e and its cluster becomes infinite at time Z_e, where $U_e < Z_e < 1$.

It is easy to see that each infinite cluster is a "tree with one end", i.e. any two infinite rays agree outside some finite set of edges. Theorem 14 gives more precise information about the infinite clusters.

4 The shape of clusters in frozen percolation

Distinguish an undirected edge $\tilde{e}$ of the binary tree $\mathbf{T} = (\mathcal{V}, \mathcal{E})$. Write $\mathcal{C}_t$ for the cluster containing $\tilde{e}$ in the state $\mathcal{A}_t$ of the frozen percolation process, with $\mathcal{C}_t$ empty if $\tilde{e} \notin \mathcal{A}_t$. Proposition 2(c) shows the probabilities of $\mathcal{C}_1$ being { infinite, finite non-empty, empty } are $\{\frac{7}{12}, \frac{1}{16}, \frac{17}{48}\}$. In this section we elaborate on the shape of $\mathcal{C}_t$.

4.1 The shape of finite clusters

In the ordinary critical percolation process $\mathcal{B}_{1/2}$ on the edges $\mathcal{E}$, write $\widetilde{\mathcal{B}}$ for the cluster containing $\tilde{e}$, conditional on edge $\tilde{e}$ being present. So $\widetilde{\mathcal{B}}$ is a certain modified critical Galton-Watson tree.

Proposition 11 *Let $\frac{1}{2} \leq t \leq 1$. Conditional on $\mathcal{C}_t$ being finite non-empty, $\mathcal{C}_t$ has the same law as $\widetilde{\mathcal{B}}$.*

Proof. Let $\mathbf{s}$ be a non-random finite subtree of $\mathbf{T}$ containing $\tilde{e}$. Write $\#\mathbf{s}$ for the number of edges of $\mathbf{s}$. Clearly

$$P(\mathbf{s} \subseteq \widetilde{\mathcal{B}}) = (1/2)^{\#\mathbf{s}-1}. \tag{12}$$

We will show

$$P(\mathbf{s} \subseteq \mathcal{C}_t \text{ and } \mathcal{C}_t \text{ is finite }) = (1/2)^{\#\mathbf{s}+3} t^{-3}. \tag{13}$$

Applying this to $\mathbf{s} = \{\tilde{e}\}$ shows

$$P(\mathcal{C}_t \text{ is finite non-empty }) = (1/2)^4 t^{-3}$$

and then applying (13) for general $\mathbf{s}$ shows

$$P(\mathbf{s} \subseteq \mathcal{C}_t | \, \mathcal{C}_t \text{ finite non-empty}) = (1/2)^{\#\mathbf{s}-1}. \tag{14}$$

Comparing (12) and (14), the desired equality of laws then follows, by inclusion-exclusion or by Dynkin's $\pi - \lambda$ lemma.

Write $\partial(\mathbf{s})$ for the set of edges in $\mathcal{E} \setminus \mathbf{s}$ which are adjacent to some edges of $\mathbf{s}$, directed away from $\mathbf{s}$. It is easy to check that $\#\partial(\mathbf{s}) = \#\mathbf{s} + 3$. Equality (13) then follows from the next lemma, since the events in (ii) are independent (Corollary 5), making the probability of event (ii) equal $(\frac{1}{2t})^{\#\mathbf{s}+3} t^{\#\mathbf{s}}$.

Lemma 12 *The following are equivalent.*
(i) $\mathbf{s} \subseteq \mathcal{C}_t$ *and* $\mathcal{C}_t$ *is finite .*
(ii) $Y_{\vec{e}} > t \ \forall \ \vec{e} \in \partial(\mathbf{s})$ *and* $U_e \leq t \ \forall e \in \mathbf{s}$.

Proof. Suppose (ii) holds. Applying (7) recursively we see that $Y_{\vec{e}} > t$ for each directing $\vec{e}$ of each edge $e \in \mathbf{s}$. Therefore each $e \in \mathbf{s}$ enters the frozen percolation process and so $\mathbf{s} \subseteq \mathcal{C}_t$. And by Proposition 6 no vertex of $\mathbf{s}$ is in an infinite cluster of $\mathcal{A}_t$, so $\mathcal{C}_t$ is finite. Conversely suppose (ii) fails, so there is some vertex $\vec{f} \in \partial(\mathbf{s})$ such that $Y_{\vec{f}} \leq t$. Now $\vec{f}$ must have a parent edge $\vec{e}$ for which the undirected edge e is in $\mathbf{s}$. If (i) holds then $e \in \mathcal{C}_t$ and so Lemma 7 implies $Y_{\vec{e}} \leq t$. Then Proposition 6 shows that some end-vertex of e is in an infinite cluster of $\mathcal{A}_t$, contradicting the assertion (i) that $\mathcal{C}_t$ is finite. □

4.2 Infinite clusters don't form at time $t = 1$

As mentioned earlier, our proof of Lemma 8 doesn't work for $t = 1$. Here is the patch needed to establish Proposition 6 for $t = 1$.

Lemma 13 $P(\#\mathcal{C}_t < \infty \ \forall t < 1, \ \#\mathcal{C}_1 = \infty) = 0.$

Proof. Because a finite cluster can only grow at times of the form U_e for some adjacent edge e,

$$P(\#\mathcal{C}_t \leq k, \#\mathcal{C}_1 = \infty) \leq P(U_{\vec{e}} \geq t \text{ for some } \vec{e} \in \mathcal{E}_{\leq k}) \leq (1-t)\#\mathcal{E}_{\leq k}$$

where $\mathcal{E}_{\leq k}$ is the set of directed edges whose distance from $\tilde{e}$ is at most k. Also,

$$P(k < \#\mathcal{C}_t < \infty) \leq P(k < \#\mathcal{C}_t < \infty | 1 \leq \#\mathcal{C}_t < \infty) = P(\#\widetilde{\mathcal{B}} > k)$$

by Proposition 11. Thus

$$\begin{aligned} P(\#\mathcal{C}_t < \infty, \#\mathcal{C}_1 = \infty) &\leq \min_k (P(\#\widetilde{\mathcal{B}} > k) + (1-t)\#\mathcal{E}_{\leq k}) \\ &\to 0 \text{ as } t \uparrow 1. \end{aligned}$$

4.3 The incipient infinite percolation cluster

As above, distinguish an undirected edge $\tilde{e}$ of the tree $\mathbf{T}$. Construct a random infinite tree $\mathcal{C}^\infty$ containing $\tilde{e}$ as follows. First pick uniformly at random an infinite ray

$$\vec{e_1}, \vec{e_2}, \vec{e_3}, \ldots, \quad \text{where } \vec{e_1} \text{ is a directing of } \tilde{e}. \tag{15}$$

Let each edge of this ray be present with probability one; let each other edge of $\mathcal{E}$ be independently present with probability $1/2$; define $\mathcal{C}^\infty$ to be the cluster of edges containing $\tilde{e}$. This $\mathcal{C}^\infty$ is the *incipient infinite percolation cluster* containing $\tilde{e}$. It arises by considering the cluster containing $\tilde{e}$ in the ordinary percolation process $\mathcal{B}_t$ with $t > 1/2$, conditioning on this cluster being infinite, and then taking a weak limit as $t \downarrow 1/2$ (cf. Kesten [10], Haase [8]). Recall $\mathcal{C}_1$ is the cluster containing $\tilde{e}$ in the final state $\mathcal{A}_1$ of the frozen percolation process. Write $Z_{\tilde{e}}$ for the time at which the cluster containing $\tilde{e}$ becomes infinite. So (Corollary 10) the event $\{\#\mathcal{C}_1 = \infty\}$ is the event $\{Z_{\tilde{e}} \leq 1\}$.

Theorem 14 *Conditional on the event $\{\#\mathcal{C}_1 = \infty\}$, $\mathcal{C}_1$ has the same law as $\mathcal{C}^\infty$ and is independent of $Z_{\tilde{e}}$.*

We find this result quite surprising. Our initial intuition was that for $t > 1/2$, the lack of availability of edges already frozen into other infinite components would mean that the trees which become infinite at t should become "thinner" as t increases.

4.4 Proof of Theorem 14

We follow the pattern of the proof of Proposition 11. Let $\mathbf{s}$ be a finite subtree of $\mathbf{T}$ containing $\tilde{e}$. Write $\#\mathbf{s}$ for the number of edges of $\mathbf{s}$. Let $\overrightarrow{e_*}$ be an edge of $\mathbf{s}$, directed, and such that no child of $\overrightarrow{e_*}$ is an edge of $\mathbf{s}$. Recall $\mathcal{C}^\infty$ is the incipient infinite percolation cluster. We claim

$$P\left(\mathbf{s} \subset \mathcal{C}^\infty \text{ and } \overrightarrow{e_*} \text{ is in the infinite directed ray (15) of } \mathcal{C}^\infty\right) = (1/2)^{\#\mathbf{s}}. \quad (16)$$

Because, writing d for the number of edges on the path from $\tilde{e}$ to $\overrightarrow{e_*}$ (including end-edges), then the chance that the ray (15) starts as this d-edge path segment equals $(1/2)^d$, and the chance that the remaining $\#\mathbf{s} - d$ edges of $\mathbf{s}$ are in $\mathcal{C}^\infty$ equals $(1/2)^{\#\mathbf{s}-d}$.

Now consider $\mathcal{C}_1$. As in the proof of Lemma 9, if $Z_{\tilde{e}} \leq 1$ there exists an infinite ray $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \ldots$ with $\overrightarrow{e_1}$ a directing of $\tilde{e}$ such that $Y_{\overrightarrow{e}} = Z_{\tilde{e}}$ for each $\overrightarrow{e}$ in the ray. Fix a finite tree $\mathbf{s}$ and an edge $\overrightarrow{e_*}$ as above. Let $1/2 < t < 1$. We shall show that the event

$$\mathbf{s} \subset \mathcal{C}_1 \text{ and } Z_{\tilde{e}} \in [t, t+dt] \text{ and } \overrightarrow{e_*} \text{ is in the infinite directed ray } \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \ldots \quad (17)$$

has probability $(\frac{1}{2})^{\#\mathbf{s}} \frac{1}{4t^4} dt$. By (16), this agrees with the probability of the corresponding event for the independent pair $(\mathcal{C}^\infty, Z_{\tilde{e}})$. Then Dynkin's $\pi - \lambda$

lemma identifies the joint laws of $(\mathcal{C}_1, Z_{\tilde{e}})$ and $(\mathcal{C}^\infty, Z_{\tilde{e}})$ as being identical on $\{Z_{\tilde{e}} \leq 1\}$, which is the assertion of Theorem 1.

We first need a criterion for when event (17) occurs. Recall from section 4.1 the set $\partial(\mathbf{s})$ of edges adjacent to $\mathbf{s}$, directed away from $\mathbf{s}$.

Lemma 15 *Event (17) occurs iff*
(i) $U_e < t$ *for each* $e \in \mathbf{s}$
(ii) $\min\{Y_{\vec{e}} : \vec{e} \in \partial(\mathbf{s})\} \in [t, t+dt]$
(iii) the minimum in (ii) is attained at some child of $\vec{e_*}$.

Proof. Suppose (i) - (iii) occur, and set $y = \min\{Y_{\vec{e}} : \vec{e} \in \partial(\mathbf{s})\} \in [t, t+dt]$. Write $\widetilde{\mathcal{E}}_{\mathbf{s}}$ for the set of directed edges corresponding to the undirected edges of $\mathbf{s}$. Consider $\vec{e} \in \widetilde{\mathcal{E}}_{\mathbf{s}}$ with both of its children outside $\widetilde{\mathcal{E}}_{\mathbf{s}}$. By the basic recursion (7), if $\vec{e} = \vec{e_*}$ then $Y_{\vec{e}} = y$ and otherwise $Y_{\vec{e}} \geq y$. Arguing recursively "away from the boundary of $\mathbf{s}$" we see that $Y_{\vec{e}} \geq y$ for those $\vec{e} \in \widetilde{\mathcal{E}}_{\mathbf{s}}$ which are not part of the ray $\vec{e_1}, \vec{e_2}, \vec{e_3}, \ldots$ and $Y_{\vec{e}} = y$ for edges $\vec{e}$ within the ray. So $Z_{\tilde{e}} = y$ and by definition (8) of $\mathcal{A}_1$ we have $\mathbf{s} \subset \mathcal{A}_1$. Conversely, suppose event (17) occurs. Then (i) is immediate. Now $Y_{\vec{e_i}} = Z_{\tilde{e}}$ for all $\vec{e_i}$ in the ray and in particular for some child of $\vec{e_*}$, so to establish (ii) and (iii) it is enough to show

$$\min\{Y_{\vec{e}} : \vec{e} \in \partial(\mathbf{s})\} \geq t.$$

If not, then $Y_{\vec{e}} = y < t$ for some $\vec{e} = (w_1, w_2) \in \partial(\mathbf{s})$. Thus the vertex w_1 enters an infinite cluster at or before time y. But $\vec{e} \in \partial(\mathbf{s})$ means w_1 is a vertex of $\mathbf{s}$ and hence of $\mathcal{C}_1$; since the infinite cluster containing w_1 can't grow after time y, this means $Z_{\tilde{e}} \leq y$, a contradiction. $\Box$

Using Lemma 15, the chance of event (17) equals

$$t^{\#\mathbf{s}} \;\times\; \#\partial(\mathbf{s}) \frac{1}{2t^2} dt \; (\tfrac{1}{2t})^{\#\partial(\mathbf{s})-1} \;\times\; \frac{2}{\#\partial(\mathbf{s})}$$

where the three terms correspond to the three requirements of the lemma. Since $\#\partial(\mathbf{s}) = \#\mathbf{s} + 3$, this reduces to $(\frac{1}{2})^{\#\mathbf{s}} \frac{1}{4t^4} dt$ as required.

5 Related work and open problems

5.1 The lattice setting

Obviously one can seek the define the frozen percolation process on $\mathbb{Z}^d$ ($d \geq 2$) or more general graphs. But proving existence and uniqueness on $\mathbb{Z}^d$ seems a challenging problem. To see the difficulty, one could start with the analogous process on the finite region $[-L, L]^d$ in which the distinction between finite and infinite clusters was replaced by the distinction between clusters disconnected from or connected to the boundary. Heuristically, letting $L \to \infty$ and taking weak limits should give the frozen percolation process on $\mathbb{Z}^d$. But a limit of finite clusters is not necessarily finite, so it seems hard to prove even existence this way.

As suggested by Jennifer Chayes and by Geoff Grimmett (personal communications), one might suspect that existence and uniqueness of the frozen percolation process on $\mathbb{Z}^d$ might be related to uniqueness of infinite clusters in supercritical percolation. The same remark holds for the next process.

5.2 A stationary process

Somewhat analogous to the frozen percolation process is the following process. Each edge of $\mathbf{T}$ may be "on" or "off". An edge which is off will turn on at (stochastic) rate 1. When an infinite cluster of "on" edges appears, all the edges in the cluster turn off simultaneously.

It seems intuitively clear that some unique stationary process satisfies this description, but I do not see a rigorous proof. Note this is a kind of interacting particle process on the edges of $\mathbf{T}$, but different from the usual processes in which only one or two changes may occur simultaneously.

5.3 Trees with one end

Random infinite trees with one end arise in several contexts. One context (Kesten [10]) is as critical or subcritical Galton-Watson branching processes, conditioned to be infinite via some limiting procedure. Aldous - Pitman [3] give a detailed study of growth processes associated with such trees. The notion of *uniform random spanning forest* on an infinite graph, analogous to uniform random spanning tree on a finite graph, has attracted study since Pemantle's [11] treatment of $\mathbb{Z}^d$. See Benjamini et al [4] for a detailed recent treatment. In many cases the tree-components of the forest have only one end, for instance ([4] Theorem 10.1) the wired uniform spanning forest on any Cayley graph which is not a finite extension of $\mathbb{Z}$, and ([4] Theorem 12.4) any planar recurrent graph with a finite number of sides to each face.

5.4 d-regular trees

Our arguments extend essentially unchanged to the tree $\mathbf{T}^d$ in which each vertex has $d \geq 3$ edges. In this setting, the random time that the cluster containing a prescribed vertex becomes infinite (given by Proposition 2(b) in the case $d = 3$) has density function

$$d(d-2)^{-1}(d-1)^{\frac{-d}{d-2}} \ x^{-\frac{2(d-1)}{d-2}}, \quad \tfrac{1}{d-1} \leq x \leq 1.$$

Writing X^d for a r.v. with the law above, we see that as $d \to \infty$ there is a limit law for dX^d with density $\frac{1}{x^2}$, $1 \leq x < \infty$. This limit law reappears below.

5.5 Random graph analogs

The Erdős - Rényi random graph process (n vertices; each edge present independently with chance t/n) provides an alternate mean-field model of percolation. The $n \to \infty$ limit of the component containing a specified vertex is the family tree of a Galton-Watson branching process with Poisson(t) offspring, for which the critical time is $t = 1$, and the analogous incipient infinite percolation cluster is this branching process conditioned to be infinite (call this law $\mathrm{PGW}^\infty(1)$, say). Now one can consider an analog of frozen percolation in the Erdős - Rényi setting, by freezing components when their size exceeds a threshold size $w(n)$ for which $w(n) \to \infty$, $w(n)/n \to 0$. Conjecture 3.6 in Aldous [1] says that in the $n \to \infty$ limit, the component ($\mathcal{C}$, say) ultimately containing a specified vertex and the time (Z, say) when the component exceeds the threshold satisfy
(i) $\mathcal{C}$ has $\mathrm{PGW}^\infty(1)$ law;
(ii) Z has density $\frac{1}{x^2}$, $1 \leq x < \infty$;
(iii) $\mathcal{C}$ and Z are independent.
This joint law is the $d \to \infty$ limit (cf. section 5.4) of the $\mathbf{T}^d$ analog of the joint law in Theorem 14. Now there is a rather subtle abstract structure (see [2] Construction 8 for an outline) which plays the role of the $n = \infty$ case of the random graph; and presumably within this structure one can construct an analog of the frozen percolation process satisfying (i)-(iii). But this construction, and deriving the weak convergence asserted in the conjecture, both seem rather tricky.

5.6 Emergence of the infinite cluster

How the cluster containing a specified edge $\tilde{e}$ becomes infinite (if it does) is qualitatively different in the frozen percolation process and the ordinary percolation process. In the latter, at some random time the finite cluster gets linked to an already-infinite cluster, whereas in the former case the cluster size $\#\mathcal{C}_t \uparrow \infty$ as $t \uparrow Z_{\tilde{e}}$. In fact, conditional on the infinite cluster $\mathcal{C}$ forming at time $Z_{\tilde{e}} < 1$, the values $(U_e,\ e \in \mathcal{C})$ are i.i.d. $U(0, Z_{\tilde{e}})$, implying that the conditioned process $(\mathcal{C}_t, 0 \leq t \leq Z_{\tilde{e}})$ is a "pruning process" in the class discussed in section 3.3 of [3].

5.7 Reconstructions and uniqueness

As mentioned in the introduction, it is not clear that the frozen percolation process $(\mathcal{A}_t, 0 \leq t \leq 1)$ is a measurable function of $(U_e, e \in \mathcal{E})$. Proving this reduces to proving that for a fixed $\overrightarrow{f} \in \overrightarrow{\mathcal{E}}$,

$$Y_{\overrightarrow{f}} \text{ is } \sigma(U_e, e \in \mathcal{E}) - \text{measurable.} \tag{18}$$

Now the process $((Y_{\overrightarrow{e}}, U_{\overrightarrow{e}}),\ \overrightarrow{e}$ a descendant of $\overrightarrow{f})$ is a branching Markov process on state-space $I \times [0,1]$ whose transition law can be written explicitly. We suspect that analysis of this branching Markov process can be used to establish (18), but we have not pursued the details. This issue is analogous to questions about the Ising model on $\mathbf{T}$, specifically about extremality of free boundary Gibbs states: see section 2.2 of Evans et al [5] for a recent account. Similarly, the question of whether the frozen percolation process constructed in Lemma 4 is the *unique* process satisfying (*) is analogous to questions about uniqueness of Gibbs distributions.

References

[1] D. Aldous. Tree-valued Markov chains and Poisson-Galton-Watson distributions. In D. Aldous and J. Propp, editors, *Microsurveys in Discrete Probability*, number 41 in DIMACS Ser. Discrete Math. Theoret. Comp. Sci, pages 1–20, 1998.

[2] D.J. Aldous. Deterministic and stochastic models for coalescence: a review of the mean-field theory for probabilists. *Bernoulli*, 5:xxx, 1999. Available via homepage http://www.stat.berkeley.edu/users/aldous.

[3] D.J. Aldous and J. Pitman. Tree-valued Markov chains derived from Galton-Watson processes. *Ann. Inst. H. Poincarè Probab. Statist.*, 34:637–686, 1998.

[4] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm. Uniform spanning forests. Unpublished, 1998.

[5] W. Evans, C. Kenyon, Y. Peres, and L.J. Schulman. Broadcasting on trees and the Ising model. Unpublished, 1998.

[6] P.J. Flory. Molecular size distribution in three dimensional polymers I. Gelation. *J. Amer. Chemical Soc.*, 63:3083–3090, 1941. See also the two following papers in that issue.

[7] G.R. Grimmett. *Percolation.* Springer-Verlag, Berlin, 1989.

[8] H. Haase. On the incipient cluster of the binary tree. *Arch. Math. (Basel)*, 63:465–471, 1994.

[9] O. Häggström. Uniform and minimal essential spanning forests on trees. *Random Structures and Algorithms*, 12:27–50, 1998.

[10] H. Kesten. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.*, 22:425–487, 1987.

[11] R. Pemantle. Choosing a spanning tree for the integer lattice uniformly. *Ann. Probab.*, 19:1559–1574, 1991.

[12] W.H. Stockmayer. Theory of molecular size distribution and gel formation in branched chain polymers. *J. Chem. Phys.*, 11:45–55, 1943.

[13] P.G.J. van Dongen. Aggregation beyond the gel point: A new class of exactly solvable models. *J. Statist. Phys.*, 87:1273–1286, 1997.