Stat206: Random Graphs and Complex Networks

Lecture 6: Proportional Attachment Model via Analytics

Lecturer: David Aldous *Scribes: Lara Dolecek*

We will first compute an approximate explicit solution of a recurrence relation. This result will come in handy later during our discussion of a PA model. In particular, we will consider a more general PA model, including the one with aging. We will discuss the expectation and degree distribution of a randomly chosen vertex at a prescribed time in the context of this model.

Background Math Result

Consider the following equation:

$y(i) = y(i-1)a(i-1) - y(i)a(i)$.

This is a recurrence relationship where $y(i)$ is not known. However, what is known is that $a(i) \sim ai$ as $i \to \infty$.

This implies that $y(i) \sim i^{-(1+\frac{1}{a})}$.

The argument for the last statement is as follows:

$\frac{y(i)}{y(i-1)} = \frac{a(i-1)}{1+a(i)} \approx \frac{a(i-1)}{1+ai} \approx \frac{ai-a}{ai+1} = \frac{ai-a}{ai(1+\frac{1}{ai})} \approx \frac{(ai-a)(1-\frac{1}{ai})}{ai} = \frac{ai-1-a+\frac{1}{i}}{ai} \approx 1 - \frac{a+1}{ai} = 1 - \frac{\frac{a+1}{a}}{i}$.

Let γ denote $\frac{a+1}{a}$, so that $\frac{y(i)}{y(i-1)} \sim 1 - \frac{\gamma}{i}$. Since $\left(\frac{i}{i-1}\right)^{-\gamma} \sim 1 - \frac{\gamma}{i}$, it follows that $y(i) \sim i^{-\gamma}$.

PA Model

Let us now consider a more general proportional attachment (PA) model that has 4 parameters:

K, L, M, and θ (study in-degrees only).

A paper [1] that treats this model is listed at the end of the notes.

In this model at each step the following occurs:

• one new vertex v_{new} is created,

• v_{new} gets K in-edges,

• L in-edges appear at uniform random vertices, and

• M in-edges appear at random vertices V, chosen according to $P(V)$, where $P(V)$ is proportional to $d(V)+\theta$ (here $d(V)$ = in-degree(V)).

After t steps there are:

• t vertices, and

• t(K+L+M) edges.

Suppose that there is a $t \to \infty$ limit distribution of the proportion of degree-i vertices, y(i), say. Thus,

$ty(i)$=E[number degree-i vertices at time t]. Let us denote this expression by $(\star)$. Now consider fixed i and step $t \to t+1$ (This corresponds to $(\star)$ for t+1 $-(\star)$ for t). Then,

$y(i) = ty(i-1)q(i-1) - ty(i)q(i)$

where

$q(i)$=P(a particular degree-i vertex gets a new in-edge in step $t \to t+1$)=$\frac{L}{t} + \frac{M(i+\theta)}{t(K+L+M)+\theta t}$.

We use the fact that the probability of rare events $\approx$ expectation.

Let $a(i) = tq(i)$ so that

$a(i) = L + \frac{M(i+\theta)}{K+L+M+\theta} \approx \frac{Mi}{K+L+M+\theta} = a \times i.$

Then,

$\gamma = \frac{a+1}{a} = \left(\frac{M}{K+L+M+\theta} + 1\right) \times \frac{K+L+M+\theta}{M} = \frac{K+L+M+\theta+M}{M} = 2 + \frac{K+L+\theta}{M}$

and $y(i)$ has tail $\approx i^{-\gamma}$(recall the calculation from the beginning).

PA Model with Aging

This model has an additional parameter $-\infty < \alpha < \infty$. The model is described as follows:

• Vertex t arrives at time t.

• This vertex gets one edge, attached to a random vertex s, which is chosen with probability proportional to $(t-s)^{-\alpha} deg(s)$.

• Random tree: could create M > 1 edges.

• If $\alpha = 0$, then it reduces to the basic PA model.

Function $f(s/t)$: Expectation of Degree of Vertex s at Time t

Let us now study E[degree of vertex s at time t] of this model. We set it equal to $f(s/t)$ for t large (guess), and seek an appropriate equation for f.

First note that E[Average degree at time t] = 2 (tree) $= \frac{1}{t}\sum_{s=1}^{t} f(s/t) = \int_0^1 f(x)dx$.

Side conditions on the function f are: $\int_0^1 f(x)dx = 2, f(x) \geq 1, f(1) = 1$.

What happens to vertex s at step $t \to t+1$?

$P(t+1 \text{ attaches to } s) \approx \frac{(t-s)^{-\alpha} f(s/t)}{\sum_{s'=1}^{t} (t-s')^{-\alpha} f(s'/t)} \frac{f(x)(1-x)^{-\alpha} t^{-\alpha}}{t\int_0^1 f(u)(1-u)^{-\alpha} t^{-\alpha} du}$

This also equals $f(\frac{s}{t+1}) - f(\frac{s}{t})\frac{-s}{t^2} f'(\frac{s}{t}) = \frac{-x}{t} f'(x)$, where we have used that $\frac{s}{t+1} - \frac{s}{t} \approx -\frac{s}{t^2}$, and $x = \frac{s}{t}$.

Equating the two expressions yields

$\frac{-f'(x)}{f(x)}x(1-x)^{\alpha} = \frac{1}{\int_0^1 f(u)(1-u)^{-\alpha}du} = \beta(\alpha) > 0.$

The last equation can be solved numerically.

Let us consider the case when $\alpha = 0$ (this corresponds to the basic PA model). Then $\frac{-f'(x)}{f(x)} = \frac{1}{2x}$ implies $f(x) = x^{-\frac{1}{2}}$.

Behavior of $f(x)$ as $x \downarrow 0$, $\alpha = 0$

Suppose first that $f(x) \sim x^{-\delta}$, for some arbitrary δ.

Then $\frac{-f'(x)}{f(x)} \times x = \frac{\delta \times x^{-\delta-1}}{x^{-\delta}} \times x$. We know that this expression is equal to β so then δ=β. Thus, $f(x)$ is actually approximately $x^{-\beta}$.

Consider a particular vertex s at time t, such that $deg(s) = i$ and random. The rate at which degree changes $i \to i+1$ is $i \times h(t)$, for some computable $h(t) = h(t, s, \alpha)$.

Degree Distribution of Random Vertex at Time t

Recall that in the Yule process there is a jump from state i to $i+1$ at the rate $i \times 1$. This implies that the state at time t has a geometric distribution ($p = e^{-t}$) and mean $\frac{1}{p} = e^t$. Rate $i \times h(t)$ holds for any deterministic $h(t)$. So the distribution at time t is $\approx$ Geom($\frac{1}{f(s/t)}$). Random vertex at time t is a vertex, say s, where $Ut = s$, and $U \sim$ Unif[0, 1].

Let D denote the degree of a random vertex at time t. Its distribution is Geom($\frac{1}{f(U)}$).

In particular, for $\alpha = 0$ the distribution of D is Geom($U^{1/2}$) (because $f(x) = x^{-\frac{1}{2}}$, as computed in the previous section).

¿From Lecture 4 we know that D is Geom($e^{-\frac{T}{2}}$) where $T \sim$ Exp(1), and $e^{-T} =_d U$.

The tail of D is like the tail of Geom(U^{β}). Matching with Lecture 4 calculation we now have that

$P(D = d) \sim d^{-(2+\frac{1-\beta}{\beta})} = d^{-\gamma}$, where $\gamma = 2 + \frac{1-\beta}{\beta}$.

References

[1] Dorogovtsev, S.N. and Mendes, J.F.F., “Evolution of Networks”, Submitted to *Advances on Physics*, March 2001.