STAT206 From Random Graphs To Complex Networks **Spring 2003**

Lecture 25: Supercritical Continuum Percolation

Lecturer: David Aldous *Scribe: Shelley Zhuang*

25.1 Supercritical Continuum Percolation (Penrose sec 10.2, 10.3)

Notation:

$\mathcal{H}_\lambda$: Poisson Point Process (PPP) with rate λ, on R^d.

$\mathcal{H}_{\lambda,s}$: restricted to $B(s) = [s/2, s/2]^d$

$\mathcal{H}_{\lambda,0}$: $\mathcal{H}_\lambda \cup \{\mathbf{0}\}$.

$\mathcal{G}(\mathcal{H}_{\lambda,s}; 1)$: graph with edges between points at distance ≤ 1.

$P_\infty(\lambda) = P($**0** is in infinite component of $\mathcal{G}(\mathcal{H}_{\lambda,0}; 1)$.

Theorem 10.9

Take $d \geq 2$ and $\lambda > \lambda_c$
L_1 = size of largest connected component of $\mathcal{G}(\mathcal{H}_{\lambda,s}; 1)$
L_2 = size of second largest component of $\mathcal{G}(\mathcal{H}_{\lambda,s}; 1)$

Then as $s \to \infty$, $\frac{L_1}{s^d} \to \lambda P_\infty(\lambda) > 0$, $\frac{L_2}{s^d} \to 0$.

With probability $\to 1$, as $s \to \infty$, the largest diameter of any small component is $O(\log s)^{1+\epsilon}$. The giant component is **crossing** for B(s).

Comment 1: Assuming $\mathcal{G}(\mathcal{H}_\lambda; 1)$ has a unique infinite component C_∞,
E(# points in $C_\infty \cap B(s)$) = $\int_{B(s)} \lambda dx P_\infty(\lambda) = s^d \lambda P_\infty(\lambda)$. Metric diameter(s) = $max_{x,y \in S} \|x - y\|$.

A 1-crossing means a crossing along the first dimension, a 2-crossing means a crossing along the second dimension, and crossing means an i-crossing, for all $1 \leq i \leq d$.

Set $B(a, j)$ = rectangle $[0, ja]$x$[0, a]$, $j = 1, 2, 4$.
Let LR_a be the event that there is a 1-crossing component of $\mathcal{G}(\mathcal{H}_\lambda \cap B(a, 2); 1)$.
Let SLR_a be the event that there is a 1-crossing component of $\mathcal{G}(\mathcal{H}_\lambda \cap B(a, 1); 1)$.
Let LLR_a be the event that there is a 1-crossing component of $\mathcal{G}(\mathcal{H}_\lambda \cap B(a, 4); 1)$.

Lemma 10.4

Take d = 2, $\lambda > \lambda_c$, then $P(LR_a)$ and $P(SLR_a) \to 1$ as $a \to \infty$.

Lemma 10.5

Take d = 2, $\lambda > \lambda_c$, then there exists $c > 0$ such that for large a, $P(LR_a) \geq 1 - e^{-ca}$.

Proposition 10.13

Take d = 2, $\lambda > \lambda_c$, and function $\phi(s) \leq s$ with $\frac{\phi(s)}{\log s} \to \infty$. Let event E_s be (i) unique component of $\mathcal{G}(\mathcal{H}_{\lambda,s}; 1)$ crossing for $B(s)$ and (ii) no other component has metric diameter $\geq \phi(s)$. Then $P(E_s \text{ fails}) \leq \exp(-\delta\phi(s))$ for some $\delta > 0$.

Set $m_s = \lceil \frac{4s}{\phi(s)} \rceil$, and $\psi_s = \frac{s}{m_s} \approx \frac{\phi(s)}{4}$.

Consider all horizontal dominos of dimension $2\psi_s$ x ψ_s, and vertical dominos of dimension ψ_s x $2\psi_s$, P(some domino has no crossing) $\leq m_s^2 e^{-c\psi_s} \leq \frac{s^2}{\phi^2(s)} e^{-c'\phi(s)} \leq e^{-c''\phi(s)}$.

How long can a line segment be and not meet any of the domino crossing paths? If points x and y are in nonadjacent squares, then any route between them must meet a domino crossing path, so $max\|x-y\| \leq 3\psi_s$. So at most 1 component of $\mathcal{G}(\mathcal{H}_{\lambda,s}; 1)$ of metric diameter $\geq (\log s)^2$.

Temporarily consider $\mathcal{G}(\mathcal{H}_{\lambda,s}; (\log s)^2)$, connect x,y if $\|x-y\| \leq (\log s)^2$
$\Rightarrow$ each small component is a clique of this graph and analyze the maximum clique size of random geometric graphs. Turns out that maximum clique size of $\mathcal{G}(\mathcal{H}_{\lambda,s}; (\log s)^2)$ is $O(\log s)^{2d}$.
$\Rightarrow$ maximum size of small component $\leq C_1(\log s)^{2d}$.

Easy ergodic arguments:
$\frac{\text{\# points in } C_\infty \cap B(s)}{s^d} \to \lambda P_\infty(\lambda)$

Let $C_1, C_2, ..., C_M$ be components of $C_\infty \cap B(s)$, enough to show $s^{-d} E \sum_{i \geq 2} |C_i| \to 0$. Each $C_i, i \geq 2$ has some point X_i in boundary layer. The balls $B(X_i, 1/2)$ are disjoint. Count volume, $M \leq C_2 s^{d-1}$. So $E \sum_i |C_i| \leq C_2 s^{d-1} C_1 (\log s)^{2d} = O(s^d)$.