Stat206: Random Graphs and Complex Networks **Spring 2003**

Lecture 2: Branching Processes

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Today we will review branching processes, including the results on extinction and survival probabilities expressed in terms of the mean and the generating function of a random variable whose distribution models the branching process. In the end we will briefly state some more advanced results.

Introduction

Let's start by considering a random variable X. If possible values of X are non negative integers, then for $p_i = P(X = i)$, the sequence $(p_i, i \geq 0)$ denotes the distribution of X. Recall that

$$E[X] = \sum_{i \geq 0} i p_i$$

Some of the standard distributions and their parameters are:

•$Binomial(n, p)$,

•$Geometric(p)$,

•$Poisson(\lambda)$,

•$Normal(\mu, \sigma^2)$, and

•$Exponential(\lambda)$.

Branching Processes

Galton-Watson Branching Process

In the Galton-Watson Branching Process (GWBP) model the parameter is a probability distribution $(p_0, p_1, ...)$, taken on by a random variable ξ. For the GWBP model we have the following rule.

Branching Rule: Each individual has a random number of children in the next generation. These random variables are independent copies of ξ and have a distribution (p_i).

Let us now review some standard results about branching processes. For simplicity let μ denote E[ξ]. Let Z_n denote the number of individuals in the nth generation. By default, we set $Z_0 = 1$, and also exclude the case when ξ is a constant.

Notice that a branching process may either become extinct or survive forever. We are interested under what conditions and with what probabilities these events occur.

Some Results on P(extinct) and P(survive)

A simple but useful result is the following.

<u>Fact 1</u>:

$$E[Z_n] = \mu^n$$

If $\mu < 1$, a consequence of this result is that

$$E\left[\sum_{n=0}^{\infty} Z_n\right] = \frac{1}{1-\mu} < \infty.$$

Since the expectation is finite,

$$P(\sum_n Z_n < \infty) = 1.$$

This implies

$$P(\text{extinct}) = 1.$$

Let $p_0 = P(\xi = 0)$ denote the probability of having no children. We can then conclude the following.

<u>Fact 2</u>:

$$\text{If } p_0 = 0 \text{ then } P(\text{survive}) = 1.$$

$$\text{If } p_0 > 0 \text{ then } P(\text{extinct}) \geq p_0 > 0.$$

GWBP and Generating Function

Another result about GWBP involves generating functions. Recall that a $\{0, 1, 2, ...\}$-valued random variable X has a generating function (GF)

$$\Phi_X(y) = \sum_{n=0}^{\infty} P(X = n) y^n.$$

<u>Fact 3</u>:

If X_1,X_2,...,X_n are n independent copies of X, then $\sum_{i=1}^{n} X_i$ has $[\Phi_X(y)]^n$ as a generating function. Proof is by induction. Generalizing this result to the case when N is random, and independent of X_i, we have that the random variable $\sum_{i=1}^{N} X_i$ has $\Phi_N(\Phi_X(y))$ as a generating function.

Therefore, in the GWBP model we are interested in, Z_n has a GF $\Phi(...\Phi(\Phi(y))...)$ (n times) where $\Phi_X(y)$ is a GF of ξ.

We can think of Z_{n+1} as a sum of Z_n copies of ξ but we can also think of it as a sum of ξ copies of Z_n.

Solution of $\rho = \Phi(\rho)$

Let us consider $\rho = P(\text{extinct})$. It can be written as

$$\rho = \sum_n P(\xi = n) \rho^n = \Phi(\rho),$$

where $\Phi(\rho)$ is the generating function of ξ. For example, $P(\text{extinct}|Z_1 = 3) = \rho^3$.

What can we say about the solution of this equation?

First note that $\rho = 1$ is always a solution of $\rho = \Phi(\rho)$, but there might also be another one. The following theorem distinguishes cases with different solutions.

<u>Theorem</u> (undergraduate):

If $\mu = E[\xi] > 1$, then the equation $\rho = \Phi(\rho)$ has a unique solution in the interval $[0, 1)$, say ρ^*, and $P(\text{extinct}) = \rho^*$. Actually one can show that if $E[\xi] = 1 + \varepsilon$ and $VAR(\xi) = \sigma^2$, then $P(\text{survive}) \approx 2\frac{\varepsilon}{\sigma^2}$.

If $\mu = E[\xi] < 1$, the only solution is $\rho = 1$ so $P(\text{extinct})$ is 1.

If $\mu = E[\xi] = 1$, then it is again the case that $P(\text{extinct}) = 1$ (we are only considering the random case where the distribution has a strictly positive variance. Technically speaking, in this case, if the variance is zero, we would have $P(\text{extinct}) = 0$). Thus $P(\sum_{n=0}^{\infty} Z_n < \infty) = 1$ but $E(\sum_{n=0}^{\infty} Z_n) = \sum_{n=0}^{\infty} 1^n = \infty$ (so the expectation can be misleading!)

Depending on the value of μ the cases are classified as:

•$\mu > 1$ is the supercritical case,

•$\mu = 1$ is the critical case, and

•$\mu < 1$ is the subcritical case.

Further Results for $\mu > 1$ Case

Consider the supercritical case, $\mu > 1$. In this case both extinction and survival are possible with non-zero probability. Let us first look at the process when conditioned on extinction.

$$P(Z_1 = i|\text{ extinct}) = \frac{P(Z_1 = i \text{ and extinct})}{P(\text{extinct})} = \frac{P(\xi = i)\rho^i}{\Phi(\rho)}.$$

Denote this last expression by $P(\hat{\xi} = i)$. One can check that $E[\hat{\xi}] < 1$. Moreover, the following holds.

<u>Fact 4:</u>

The distribution of the GWBP(ξ), conditioned on the extinction is the same as the distribution of the GWBP($\hat{\xi}$).

Notice that the former is a priori a supercritical case, and that the latter is a subcritical case.

Let us now consider the case when the process survives forever. Here we can differentiate between two kinds of nodes, those that have descendants forever and those that do not. What can we say about Z_n under this scenario?

First recognize that $P(Z_n = 0) \to \rho$ as $n \to \infty$. Using $E(X|A) = E(X1_A)/P(A)$, we have that

$$E(Z_n|\text{not extinct at n}) \sim \frac{\mu^n}{1-\rho} \text{ as n} \to \infty.$$

We will now state two more results regarding the supercritical case.

Theorem (Graduate)

For $1 < \mu < \infty$, the following holds:

$$\frac{Z_n}{\mu^n} \to_{a.s} W \text{ as n} \to \infty,$$

for some positive random variable W. (uses Martingale Limit Theorem)

Theorem (Post-Graduate)

If $E(\xi \log^+ \xi) < \infty$, then

$$E\left[\frac{Z_n}{\mu^n}\right] \to E(W) = 1, \text{ and } P(W > 0) = 1 - \rho.$$

Bottom line in words: In the supercritical case either the process becomes extinct or Z_n grows as $W\mu^n$ for a random W, where $W > 0$. W is a function of the first generation, and this is known as the Founder Effect.

Probability of Siblings of a Randomly Chosen Child

Let us now consider X as the number of children in a randomly picked family, and Y as the number of siblings in the family of a randomly picked child. Then

$$P(Y = i) = \frac{(i+1)P(X = i+1)}{\mu},$$

where μ denotes $E[X]$.

The reasoning is that if there are N families then on average there are μN children. The number of children in $(i+1)$-child families is then equal to $N \cdot P(X = i+1) \cdot (i+1)$. Thus

$$P(Y = i) = \frac{N \cdot P(X = i+1) \cdot (i+1)}{N \cdot \mu},$$

and the result follows.

Remark: X and Y have the same distribution if and only if it is $Poisson(\lambda)$, for $0 < \lambda < \infty$.