Directed Scale-Free Graphs

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Abstract

We introduce a model for directed scale-free graphs that grow with preferential attachment depending in a natural way on the in- and out-degrees. We show that the resulting in- and out-degree distributions are power laws with different exponents, reproducing observed properties of the world-wide web. We also derive exponents for the distribution of in- (out-) degrees among vertices with fixed out- (in-) degree. We conclude by suggesting a corresponding model with hidden variables.

1 Introduction

Recently many new random graph models have been introduced and analyzed, inspired by certain common features observed in many large-scale real-world graphs such as the 'web graph', whose vertices are web pages (or sites), with a directed edge for each link between two web pages. For an overview see the survey papers [2] and [15]. Other graphs studied are the 'internet graph' [18], movie actor [28] and scientific [25] collaboration graphs, cellular networks [21] and many other examples.

In addition to the 'small-world phenomenon' of logarithmic diameter investigated originally by Strogatz and Watts [28], one of the main observations is that the graphs are 'scale-free' (see [5, 7, 24] and the references therein); the distribution of vertex degrees follows a power law, rather than the Poisson distribution of the classical random graph models $G(n,p)$ and $G(n,M)$ [16, 17, 19], see also [9].

Many models have been suggested to explain this and other features of the graphs studied. One of the basic ideas is the combination of growth with 'preferential attachment'; the graph grows one vertex at a time, and edges are added, perhaps only from the new vertex to old vertices, or perhaps

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also between old vertices, where the old vertices involved are chosen with probabilities proportional to their degrees. One of the simplest and earliest models is that outlined by Barabási and Albert in [5], made precise in [11]. The degree sequence of this model was analyzed heuristically in [5, 6], and rigorously in [12]. Many generalizations have been suggested and studied heuristically; a few have been analyzed precisely, see [27]). In a complicated paper Cooper and Frieze [14] have analyzed rigorously a very general version of the model allowing for (finite) distributions of out-degrees and mixtures of uniform and preferential attachment.

The models mentioned above essentially describe undirected graphs. The only exception is [14], where the authors treat either in-degree or out-degrees, but not both simultaneously; a full treatment of directed graphs was announced there, but has not yet appeared. However, in many contexts – for example the web graph – it is natural to look at directed graphs, and to study the (often different) power laws for in- and out-degrees. Here we propose a very natural model of directed web graphs and show that it gives power laws consistent with those that have been observed in the WWW.

Before turning to our model let us briefly mention two rather different kinds of model: Newman, Strogatz and Watts [26], Aiello, Chung and Lu [1] and other groups have studied random graphs chosen by first fixing the (scale-free) degree distribution, and then choosing a graph with this degree distribution. This is very different from our aim here, which is to *explain* the power-law distributions. Also, instead of preferential attachment, copying models have been studied [22, 24]; for the web this is very natural, and something like this is needed to explain the high density of small subgraphs. Again, however, such models are not what we are concerned with here: firstly they do not model out-degree distributions (the out-degrees are fixed). Secondly, they are rather specific. By keeping the model simple we hope that it can give insight into many different scale-free graphs, rather than just the web graph. Also, Cooper and Frieze [14] note that for (in-)degree distribution there is little difference between copying and preferential attachment.

2 The model

We consider a graph which grows by adding single edges at discrete time steps. At each such step a vertex may or may not also be added. For simplicity we allow multiple edges and loops. More precisely, let α, β, γ, δ_{in} and δ_{out} be non-negative real numbers, with $\alpha+\beta+\gamma=1$. Let G_0 be any fixed initial graph, for example a single vertex without edges, and let t_0 be the number of edges of G_0. (Depending on the parameters, we may have to assume $t_0 \geq 1$ for the first few steps of our process to make sense.) We set $G(t_0) = G_0$, so at time t the graph $G(t)$ has exactly t edges, and a random number $n(t)$ of vertices. In what follows, *to choose* a vertex v of $G(t)$ *according to* $d_{out} + \delta_{out}$ means to choose v so that $\Pr(v = v_i)$ is proportional to $d_{out}(v_i) + \delta_{out}$, i.e., so that $\Pr(v = v_i) = (d_{out}(v_i) + \delta_{out})/(t + \delta_{out}n(t))$. To *choose* v *according to* $d_{in} + \delta_{in}$ means to choose v so that $\Pr(v = v_i) = (d_{in}(v_i) + \delta_{in})/(t + \delta_{in}n(t))$, where all degrees are measured in $G(t)$.

For $t \geq t_0$ we form $G(t+1)$ from $G(t)$ according the the following rules:

(A) With probability α, add a new vertex v together with an edge from v to an existing vertex w, where w is chosen according to $d_{in} + \delta_{in}$.

(B) With probability β, add an edge from an existing vertex v to an existing vertex w, where v and w are chosen independently, v according to $d_{out} + \delta_{out}$, and w according to $d_{in} + \delta_{in}$.

(C) With probability γ, add a new vertex w and an edge from an existing vertex v to w, where v is chosen according to $d_{out} + \delta_{out}$.

The probabilities α, β and γ clearly should add up to one. To avoid trivialities, we will also assume that $\alpha + \gamma > 0$. When considering the web graph we take $\delta_{out} = 0$; the motivation is that vertices added under step (C) correspond to web pages which purely provide content - such pages never change, are born without out-links and remain without out-links. Vertices added under step (A) correspond to usual pages, to which links may be later added. While mathematically it seems natural to take $\delta_{in} = 0$ in addition to $\delta_{out} = 0$, this gives a model in which every page not in G_0 has either no in-links or no out-links, which is rather unrealistic and uninteresting! A non-zero value of δ_{in} corresponds to insisting that a page is not considered part of the web until something points to it, typically one of the big search engines. It is natural to consider these edges from search engines separately from the rest of the graph, as they are of a rather different nature; for the same reason it is natural not to insist that δ_{in} be an integer. We include the parameter δ_{out} to make the model symmetric with respect to reversing the directions of edges (swapping α with γ and δ_{in} with δ_{out}), and because we expect the model to be applicable in contexts other than that of the web graph.

Our model allows loops and multiple edges; there seems no reason to exclude them. However, there will not be very many, so excluding them would not significantly affect our conclusions.

Note also that our model includes (a precise version of) the $m = 1$ case of the original model of Barabási and Albert as a special case, taking $\beta = \gamma = \delta_{out} = 0$ and $\alpha = \delta_{in} = 1$. We could introduce more parameters, adding m edges for each new vertex, or (as in [14]) a random number with a certain distribution, but one of our aims is to keep the model simple, and the main effect, of varying the overall average degree, can be achieved by varying β.

3 Analysis

Having decided on the model it is not hard to find the power laws for in- and out-degrees. Throughout we fix constants $\alpha, \beta, \gamma \geq 0$ summing to 1 and $\delta_{in}, \delta_{out} \geq 0$, and set

$$c_1 = \frac{\alpha + \beta}{1 + \delta_{in}(\alpha + \gamma)} \quad \text{and} \quad c_2 = \frac{\beta + \gamma}{1 + \delta_{out}(\alpha + \gamma)}.$$

We also fix a positive integer t_0 and an initial graph $G(t_0)$ with t_0 edges. Let us write $x_i(t)$ for the number of vertices of $G(t)$ with in-degree i, and $y_i(t)$ for the number with out-degree i.

Note that the in-degree distribution becomes trivial if $\alpha\delta_{in} + \gamma = 0$ (all vertices not in G_0 will have in-degree zero) or if $\gamma = 1$ (all vertices not in G_0 will have in-degree 1), while for $\gamma\delta_{out} + \alpha = 0$ or $\alpha = 1$ the out-degree distribution becomes trivial. We will therefore exclude these cases in the following theorem.

Theorem 1. *Let $i \geq 0$ be fixed. There are constants p_i and q_i such that $x_i(t) = p_i t + o(t)$ and $y_i(t) = q_i t + o(t)$ hold with probability 1. Furthermore, if $\alpha\delta_{in} + \gamma > 0$ and $\gamma < 1$, then as $i \to \infty$ we have*

$$p_i \sim C_{IN} i^{-X_{IN}},$$

where $X_{IN} = 1 + 1/c_1$ and C_{IN} is a positive constant. If $\gamma\delta_{out} + \alpha > 0$ and $\alpha < 1$, then as $i \to \infty$ we have

$$q_i \sim C_{OUT} i^{-X_{OUT}},$$

with $X_{OUT} = 1 + 1/c_2$ and C_{OUT} is a positive constant.

In the statement above, the $o(t)$ notation refers to $t \to \infty$ with i fixed, while $a(i) \sim b(i)$ means $a(i)/b(i) \to 1$ as $i \to \infty$.

Proof. Note first that if the initial graph has n_0 vertices then $n(t)$ is equal to n_0 plus a Binomial distribution with mean $(\alpha+\gamma)(t-t_0)$. It follows from standard results (e.g., the Chernoff bound) that there is a positive constant c such that for all sufficiently large t we have

$$\Pr\left(\left|n(t)-(\alpha+\gamma)t\right| \geq t^{1/2}\log t\right) \leq e^{-c(\log t)^2}. \tag{1}$$

In particular, the probability above is $o(t^{-1})$ as $t\to\infty$.

We consider how the vector $(x_0(t), x_1(t), \ldots)$, giving for each i the number of vertices of in-degree i in the graph $G(t)$, changes as t increases by 1. Let $G(t)$ be given. Then with probability α a new vertex with in-degree 0 is created at the next step, and with probability γ a new vertex with in-degree 1 is created. More importantly, with probability $\alpha+\beta$ the in-degree of an old vertex is increased. In going from $G(t)$ to $G(t+1)$, from the preferential attachment rule, given that we perform operation (A) or (B), the probability that a particular vertex of in-degree i has its in-degree increased is exactly $(i+\delta_{in})/(t+\delta_{in}n(t))$. Since the chance that we perform (A) or (B) is $\alpha+\beta$, and since $G(t)$ has exactly $x_i(t)$ vertices of in-degree i, the chance that one of these becomes a vertex of in-degree $i+1$ in $G(t+1)$ is exactly

$$(\alpha+\beta)x_i(t)\frac{i+\delta_{in}}{t+\delta_{in}n(t)},$$

so with this probability the number of vertices of in-degree i decreases by 1. However, with probability

$$(\alpha+\beta)x_{i-1}(t)\frac{i-1+\delta_{in}}{t+\delta_{in}n(t)}$$

a vertex of in-degree $i-1$ in $G(t)$ becomes a vertex of in-degree i in $G(t)$, increasing the number of vertices of in-degree i by 1. Putting these effects together,

$$\mathbf{E}\Big(x_i(t+1)\Big|G(t)\Big) = x_i(t) + \frac{\alpha+\beta}{t+\delta_{in}n(t)}\Big((i-1+\delta_{in})x_{i-1}(t) - (i+\delta_{in})x_i(t)\Big) + \alpha\mathbf{1}_{\{i=0\}} + \gamma\mathbf{1}_{\{i=1\}}, \tag{2}$$

where we take $x_{-1}(t)=0$, and write $\mathbf{1}_{\mathcal{A}}$ for the indicator function which is 1 if the event $\mathcal{A}$ holds and 0 otherwise.

Let i be fixed. We wish to take the expectation of both sides of (2). The only problem is with $n(t)$ in the second term on the right hand side. For this, note that from a very weak form of (1), with probability $1-o(t^{-1})$ we have $|n(t)-(\alpha+\gamma)t| = o(t^{3/5})$. Now whatever value $n(t)$ takes we have

$$\frac{\alpha+\beta}{t+\delta_{in}n(t)}(j+\delta_{in})x_j(t) = O(1)$$

for each j, so

$$\mathbf{E}\left(\frac{\alpha+\beta}{t+\delta_{in}n(t)}(j+\delta_{in})x_j(t)\right) = \frac{\alpha+\beta}{t+\delta_{in}(\alpha+\gamma)t}(j+\delta_{in})\mathbf{E}x_j(t) + o(t^{-2/5})$$

and, taking the expectation of both sides of (2),

$$\mathbf{E}x_i(t+1) = \mathbf{E}x_i(t) + \frac{\alpha+\beta}{t+\delta_{in}(\alpha+\gamma)t}\Big((i-1+\delta_{in})\mathbf{E}x_{i-1}(t) - (i+\delta_{in})\mathbf{E}x_i(t)\Big) + \alpha\mathbf{1}_{\{i=0\}} + \gamma\mathbf{1}_{\{i=1\}} + o(t^{-2/5}).$$

Let us write $\overline{x_i}(t)$ for the 'normalized expectation' $\mathbf{E}x_i(t)/t$. Recalling that $c_1 = (\alpha+\beta)/(1+\delta_{in}(\alpha+\gamma))$, we have

$$(t+1)\overline{x_i}(t+1) - t\overline{x_i}(t) = c_1\Big((i-1+\delta_{in})\overline{x_{i-1}}(t) - (i+\delta_{in})\overline{x_i}(t)\Big) \\ + \alpha\mathbf{1}_{\{i=0\}} + \gamma\mathbf{1}_{\{i=1\}} + o(t^{-2/5}). \quad (3)$$

Now let $p_{-1} = 0$ and for $i \geq 0$ define p_i by

$$p_i = c_1\Big((i-1+\delta_{in})p_{i-1} - (i+\delta_{in})p_i\Big) + \alpha\mathbf{1}_{\{i=0\}} + \gamma\mathbf{1}_{\{i=1\}}. \quad (4)$$

Our first claim is that for each i we have

$$\mathbf{E}(x_i(t)) = p_i t + o(t^{3/5}) \quad (5)$$

as $t \to \infty$; later we shall show that $x_i(t)$ is concentrated around its mean, and then finally that the p_i follow the stated power law. To see (5), set $\epsilon_i(t) = \overline{x_i}(t) - p_i$. Then subtracting (4) from (3),

$$(t+1)\epsilon_i(t+1) - t\epsilon_i(t) = c_1(i-1+\delta_{in})\epsilon_{i-1}(t) - c_1(i+\delta_{in})\epsilon_i(t) + o(t^{-2/5}),$$

which we can rewrite as

$$\epsilon_i(t+1) = \frac{t - c_1(i+\delta_{in})}{t+1}\epsilon_i(t) + \Delta_i(t), \quad (6)$$

where $\Delta_i(t) = c_1(i-1+\delta_{in})\epsilon_{i-1}(t)/(t+1) + o(t^{-7/5})$.

To prove (5) we must show exactly that $\epsilon_i(t) = o(t^{-2/5})$ for each i. We do this by induction on i; suppose that $i \geq 0$ and $\epsilon_{i-1}(t) = o(t^{-2/5})$, noting that $\epsilon_{-1}(t) = 0$, so the induction starts. Then $\Delta_i(t) = o(t^{-7/5})$, and from (6) one can check (for example by solving this equation explicitly for $\epsilon_i(t)$ in terms of $\Delta_i(t)$) that $\epsilon_i(t) = o(t^{-2/5})$. This completes the proof of (5).

Our next aim is to show that, with probability 1, we have

$$x_i(t)/t \to p_i, \quad (7)$$

as claimed in the statement of the theorem. To do this we show concentration of $x_i(t)$ around its expectation using, as usual, the Azuma-Hoeffding inequality [4, 20] (see also [10]). This can be stated in the following form: if X is a random variable determined by a sequence of n choices, and changing one choice changes the value of X by at most θ, then

$$\Pr\big(|X - \mathbf{E}X| \geq x\big) \leq 2e^{-\frac{x^2}{2n\theta^2}}. \quad (8)$$

To apply this let us first choose for each time step which operation (A), (B) or (C) to perform. Let $\mathcal{A}$ be an event corresponding to one (infinite) sequence of such choices. Note that for almost all $\mathcal{A}$ (in the technical sense of probability 1), the argument proving (5) actually gives

$$\mathbf{E}(x_i(t) \mid \mathcal{A}) = p_i t + o(t). \quad (9)$$

(We leave out the straightforward but somewhat technical details.)

Given $\mathcal{A}$, to determine $G(t)$ it remains to choose at each step which old vertex (for (A) or (C)), or which old vertices (for (B)) are involved. There are at most $2t$ old vertex choices to make. Changing one of these choices from v to v', say, only affects the degrees of v and v' in the final graph. (To preserve proportional attachment at later stages we must redistribute later edges among v and v' suitably, but no other vertex is affected.) Thus $x_i(t)$ changes by at most 2, and, applying (8), we have

$$\Pr\left(\left|x_i(t) - \mathbf{E}\left(x_i(t) \mid \mathcal{A}\right)\right| \geq t^{3/4} \mid \mathcal{A}\right) \leq 2e^{-\sqrt{t}/16}.$$

Together with (9) this implies that (7) holds with probability one, proving the first part of the theorem. (Note that with a little more care we can probably replace (7) with $x_i(t) = p_i t + O(t^{1/2}\log t)$. Certainly our argument gives an error bound of this form in (5); the weaker bound stated resulted from replacing $t^{1/2}\log t$ in (1) by $o(t^{3/5})$ to simplify the equations. However the technical details leading to (9) may become complicated if we aim for such a tight error bound.)

We now turn to the more substantial part of the result, determining the behaviour of the quantities p_i defined by (4).

Assuming $\gamma < 1$, we have $\alpha + \beta > 0$ and hence $c_1 > 0$, so we can rewrite (4) as

$$(i + \delta_{in} + c_1^{-1})p_i = (i - 1 + \delta_{in})p_{i-1} + c_1^{-1}(\alpha \mathbf{1}_{\{i=0\}} + \gamma \mathbf{1}_{\{i=1\}}).$$

This gives $p_0 = \alpha/(1 + c_1\delta_{in})$, $p_1 = (1 + \delta_{in} + c_1^{-1})^{-1}(\frac{\alpha\delta_{in}}{1+c_1\delta_{in}} + \frac{\gamma}{c_1})$ and, for $i \geq 1$,

$$p_i = \frac{(i-1+\delta_{in})_{i-1}}{(i+\delta_{in}+c_1^{-1})_{i-1}} p_1 = \frac{(i-1+\delta_{in})!}{(i+\delta_{in}+c_1^{-1})!} \frac{(1+\delta_{in}+c_1^{-1})!}{\delta_{in}!} p_1. \tag{10}$$

Here, as usual, for x a real number and n an integer we write $(x)_n$ for $x(x-1)\cdots(x-n+1)$. Also, we use $x!$ for $\Gamma(x+1)$ even if x is not an integer. Also, we skip some detail in the derivations, as equations such as (4) clearly have unique solutions, and it is straightforward to check that the formulae we obtain do indeed give solutions. One can check that, as expected, $\sum_{i=0}^{\infty} p_i = \alpha + \gamma$; there are $(\alpha + \gamma + o(1))t$ vertices at large times t.

From (10) we see that as $i \to \infty$ we have $p_i \sim C_{IN} i^{-x_{IN}}$ with

$$x_{IN} = (\delta_{in} + c_1^{-1}) - (-1 + \delta_{in}) = 1 + 1/c_1,$$

as claimed.

For out-degrees the calculation is exactly the same after interchanging the roles of α and γ and of δ_{in} and δ_{out}. Under this interchange c_1 becomes c_2, so the exponent in the power law for out-degrees is $x_{OUT} = 1 + 1/c_2$, as claimed. □

We now turn to more detailed results, considering in- and out-degree at the same time. Let $n_{ij}(t)$ be the number of vertices of $G(t)$ with in-degree i and out-degree j.

Theorem 2. *Assume the conditions of Theorem 1 hold, that $\alpha, \gamma < 1$, and that $\alpha\delta_{in} + \gamma\delta_{out} > 0$. Let $i, j \geq 0$ be fixed. Then there is a constant f_{ij} such that $n_{ij}(t) = f_{ij}t + o(t)$ holds with probability 1. Furthermore, for $j \geq 1$ fixed and $i \to \infty$,*

$$f_{ij} \sim C_j\, i^{-X'_{IN}}, \tag{11}$$

while for $i \geq 1$ fixed and $j \to \infty$,

$$f_{ij} \sim D_i\, j^{-X'_{OUT}}, \tag{12}$$

where the C_j and D_i are positive constants,

$$X'_{IN} = 1 + 1/c_1 + c_2/c_1(\delta_{out} + \mathbf{1}_{\{\gamma\delta_{out}=0\}})$$

and

$$X'_{OUT} = 1 + 1/c_2 + c_1/c_2(\delta_{in} + \mathbf{1}_{\{\alpha\delta_{in}=0\}}).$$

The proof of this theorem is given in the appendix.

Before discussing the application of Theorems 1 and 2 to the web graph, note that if $\delta_{out} = 0$ (as we shall assume when modelling the web graph), vertices born with out-degree 0 always have out-degree 0. Such vertices exist only if $\gamma > 0$. Thus $\gamma\delta_{out} > 0$ is exactly the condition needed for the graph to contain vertices with non-zero out-degree which were born with out-degree 0. It turns out that when such vertices exist they dominate the behaviour of f_{ij} for $j > 0$ fixed and $i \to \infty$. A similar comment applies to $\alpha\delta_{in}$ with in- and out-degrees interchanged. If $\alpha\delta_{in} = \gamma\delta_{out} = 0$ then every vertex not in G_0 will have either in- or out-degree 0.

Note also for completeness that if $\gamma\delta_{out} > 0$ then (11) holds for $j = 0$ also. If $\gamma = 0$ then $f_{i0} = 0$ for all i. If $\gamma > 0$ but $\delta_{out} = 0$, then among vertices with out-degree 0 (those born at a type (C) step), the evolution of in-degree is the same as among those with non-zero out-degree. It follows from Theorem 1 that in this case $f_{i0} \sim C_0 i^{-X_{IN}}$.

4 Particular values

An interesting question is for which parameters (if any) our model reproduces the observed power laws for certain real-world graphs, in particular, the web graph.

For this section we take $\delta_{out} = 0$ for the reasons explained in section 2. We assume that $\alpha > 0$, as otherwise there will only be finitely many vertices (the initial ones) with non-zero out-degree. As before, let $c_1 = (\alpha + \beta)/(1 + \delta_{in}(\alpha + \gamma))$ and note that now $c_2 = 1 - \alpha$. We have shown that the power-law exponents are

$$x_{IN} = 1 + 1/c_1$$

for in-degree overall (or in-degree with out-degree fixed as 0),

$$x_{OUT} = 1 + 1/c_2$$

for out-degree overall, and that if $\delta_{in} > 0$ we have exponents

$$x'_{IN} = 1 + 1/c_1 + c_2/c_1$$

for in-degree among vertices with fixed out-degree $j \geq 1$, and

$$x'_{OUT} = 1 + 1/c_2 + \delta_{in} c_1/c_2$$

for out-degree among vertices with fixed in-degree $i \geq 0$.

For the web graph recently measured values of the first two exponents [13] are $x_{IN} = 2.1$ and $x_{OUT} = 2.7$. (Earlier measurements in [3] and [23] gave the same value for x_{IN} but smaller values for x_{OUT}.) Our model gives these exponents if and only if $c_2 = .59$, so $\alpha = .41$, and $c_1 = 1/1.1$, so

$$\delta_{in} = \frac{1.1(\alpha + \beta) - 1}{1 - \beta}.$$

This equation gives a range of solutions: the extreme points are $\delta_{in} = 0, \beta = .49, \gamma = .1$ and $\delta_{in} = .24, \beta = .59, \gamma = 0$.

As a test of the model one could measure the exponents x'_{IN} and x'_{OUT} (which may of course actually vary when the fixed out-/in-degree is varied). We obtain 2.75 for x'_{IN} and anything in the interval $[2.7, 3.06]$ for x'_{OUT}.

5 Other models

In many contexts, such as the web graph, it is clear that while preferential attachment is important, it is not the only, or perhaps even the main, reason for widely varying degrees. Another underlying cause which can produce this effect is the varying *fitness* or *attractiveness* of vertices or web pages; some web pages are just more interesting than others. This can be modelled mathematically using 'hidden variables'; each vertex has a random attractiveness, and preferential attachment depends on this and on degree. A model along these lines has been proposed by Bianconi and Barabási [8] (see also [15]).

Here we would like to propose a corresponding model for directed graphs: when a vertex v is created, two random numbers are associated with it, λ_v and μ_v, its *in-* and *out-fitness*. Let us fix two distributions D_{IN} and D_{OUT} on the non-negative real numbers. (The simplest examples would be exponential or power-law distributions.) In what follows, for each new vertex v created we choose independently λ_v from D_{IN} and μ_v from D_{OUT}, these choices being independent of all earlier choices. As before we fix $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma = 1$, and also $\delta_{in}, \delta_{out} \geq 0$. At time $t_0 \geq 0$ we start with an initial graph G_0 with t_0 edges and $n_0 \geq 1$ vertices, with certain fitnesses λ_v, μ_v for the vertices v of G_0.

For $t \geq t_0$ we form $G(t+1)$ from $G(t)$ as follows:

(A) With probability α, add a new vertex v together with an edge from v to an existing vertex w, where w is chosen according to $\lambda(d_{in} + \delta_{in})$.

(B) With probability β, add an edge from an existing vertex v to an existing vertex w, where v and w are chosen independently, v according to $\mu(d_{out} + \delta_{out})$, and w according to $\lambda(d_{in} + \delta_{in})$.

(C) With probability γ, add a new vertex w and an edge from an existing vertex v to w, where v is chosen according to $\mu(d_{out} + \delta_{out})$.

Here, to *choose v according to $\lambda(d_{in}+\delta_{in})$* means to choose v so that $\Pr(v = v_i)$ is proportional to $\lambda_{v_i}(d_{in}(v_i) + \delta_{in})$, and to *choose v according to $\mu(d_{out} + \delta_{out})$* means to choose v so that $\Pr(v = v_i)$ is proportional to $\mu_{v_i}(d_{out}(v_i) + \delta_{out})$, where the degrees are measured in $G(t)$.

Since the in- and out-degrees of vertices with different fitness will grow at different power-law rates, this model will produce some vertices of very high in-degree but low out-degree and vice-versa. This will be the topic of a forthcoming paper. Of course, one could also consider more general 'preference functions', depending on attractiveness and degree in a more complicated way, as well as a joint distribution for λ_v, μ_v, combined with extra parameters as in [14]. However, there is always some benefit in keeping the model simple and the number of parameters small.

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Appendix: Proof of Theorem 2

In this appendix, we give the proof of Theorem 2. Arguing as in the proof of Theorem 1 we see that for each i and j we have $n_{ij}(t)/t \to f_{ij}$, where the f_{ij} satisfy

$$\begin{aligned} f_{ij} \quad = \quad & c_1(i-1+\delta_{in})f_{i-1,j} - c_1(i+\delta_{in})f_{ij} \\ & +c_2(j-1+\delta_{out})f_{i,j-1} - c_2(j+\delta_{out})f_{ij} \\ & +\alpha\mathbf{1}_{\{i=0,j=1\}} + \gamma\mathbf{1}_{\{i=1,j=0\}}. \end{aligned} \tag{13}$$

Of course we take f_{ij} to be zero if i or j is -1. (At first sight there might seem to be a problem caused by the possibility that a vertex sends a loop to itself, increasing both its in- and out-degrees in one step. While this does complicate the equations for $\mathbf{E}(n_{ij}(t))$, it is easy to see that for fixed i and j the effect on this expectation is $o(t)$, so (13) holds exactly.)

We start by finding an expansion for f_{ij} when $i \to \infty$ with j fixed.

The recurrence relation (13) is of the form

$$\mathcal{L}(\mathbf{f}) = \alpha\mathbf{1}_{\{i=0,j=1\}} + \gamma\mathbf{1}_{\{i=1,j=0\}}$$

for a linear operator $\mathcal{L}$ on the two-dimensional array of coefficients f_{ij}. It is clear from the form of $\mathcal{L}$ that there is a unique solution to this equation. By linearity we can write

$$f_{ij} = g_{ij} + h_{ij}$$

where

$$\mathcal{L}(\mathbf{g}) = \alpha\mathbf{1}_{\{i=0,j=1\}} \tag{14}$$

and

$$\mathcal{L}(\mathbf{h}) = \gamma\mathbf{1}_{\{i=1,j=0\}}. \tag{15}$$

Let us first consider $\mathbf{g}$. As $\alpha, \gamma < 1$ we have $c_1, c_2 > 0$, so setting

$$b_j = \delta_{in} + \frac{1}{c_1} + \frac{c_2}{c_1}(j+\delta_{out}),$$

dividing (14) through by c_1 we obtain

$$(i+b_j)g_{ij} = (i-1+\delta_{in})g_{i-1,j} + \frac{c_2(j-1+\delta_{out})}{c_1}g_{i,j-1} + \frac{\alpha}{c_1}\mathbf{1}_{\{i=0,j=1\}}. \tag{16}$$

Using (16), it is not hard to show that $g_{ij} = 0$ for all $i > 0$ if $\alpha\delta_{in} = 0$. For the moment, we therefore shall assume that $\alpha\delta_{in} > 0$.

Note that, from the boundary condition, we have $g_{i0} = 0$ for all i. Thus for $j = 1$ the second term on the right of (16) disappears, and we see (skipping the details of the algebra) that

$$g_{i1} = a\frac{(i-1+\delta_{in})!}{(i+b_1)!}$$

where

$$a = \alpha\frac{(b_1-1)!}{c_1(\delta_{in}-1)!}$$

is a positive constant. (Here we have used $\alpha\delta_{in} > 0$.)

For $j \geq 2$ the last term in (16) is always zero. Solving for g_{ij} by iteration, we get

$$g_{ij} = \frac{c_2(j-1+\delta_{out})}{c_1} \sum_{k=0}^{i} \frac{(i-1+\delta_{in})_{i-k}}{(i+b_j)_{i-k+1}} g_{k,j-1}. \tag{17}$$

Suppose that for some constants A_{jr} we have

$$g_{ij} = \sum_{r=1}^{j} A_{jr} \frac{(i-1+\delta_{in})!}{(i+b_r)!} \tag{18}$$

for all $1 \leq j \leq j_0$ and all $i \geq 0$. Note that we have shown this for $j_0 = 1$, with $A_{11} = a$. Let $j = j_0 + 1$. Then, using (17) and (18), we see that

$$g_{ij} = \sum_{r=1}^{j-1} \frac{c_2(j-1+\delta_{out})}{c_1} A_{j-1,r} \sum_{k=0}^{i} \frac{(i-1+\delta_{in})!}{(i+b_j)_{i-k+1}(k+b_r)!}. \tag{19}$$

Now it is straightforward to verify that if $0 < y < x$ and s is an integer with $0 \leq s \leq i+1$, then

$$\sum_{k=s}^{i} \frac{1}{(i+x)_{i-k+1}(k+y)!} = \frac{1}{x-y}\left(\frac{1}{(i+y)!} - \frac{(s-1+x)!}{(i+x)!(s-1+y)!}\right). \tag{20}$$

(For example one can use downwards induction on s starting from $s = i+1$ where both sides are zero.) Combining (19) and the $s = 0$ case of (20) we see that

$$g_{ij} = \sum_{r=1}^{j-1} \frac{c_2(j-1+\delta_{out})}{c_1} A_{j-1,r} \frac{(i-1+\delta_{in})!}{b_j - b_r}\left(\frac{1}{(i+b_r)!} - \frac{(b_j-1)!}{(i+b_j)!(b_r-1)!}\right).$$

Collecting coefficients of $1/(i+b_r)!$ for different values of r, and noting that $b_j - b_r = (j-r)c_2/c_1$, we see that (18) holds for $j = j_0 + 1$, provided that

$$A_{jr} = \frac{j-1+\delta_{out}}{j-r} A_{j-1,r}$$

for $1 \leq r \leq j-1$, and

$$A_{jj} = -\sum_{r=1}^{j-1} \frac{j-1+\delta_{out}}{j-r} \frac{(b_j-1)!}{(b_r-1)!} A_{j-1,r}.$$

In fact we have the power law we are interested in (for $\mathbf{g}$ rather than $\mathbf{f}$) without calculating the A_{jr}. Observing only that $A_{11} > 0$, so $A_{j1} > 0$ for every $j \geq 1$, the $r = 1$ term dominates (18). Thus for any fixed $j > 0$ we have

$$g_{ij} \sim C'_j i^{-1+\delta_{in}-b_1} = C'_j i^{-(1+1/c_1+c_2/c_1(1+\delta_{out}))}. \tag{21}$$

Having said that we do not need the A_{jr} for the power law, we include their calculation for completeness since it is straightforward. Skipping the rather unpleasant derivation, we claim that

$$A_{jr} = a(-1)^{r-1} \frac{(j-1+\delta_{out})!}{\delta_{out}!(j-1)!} \binom{j-1}{r-1} \frac{(b_r-1)!}{(b_1-1)!},$$

for the same constant a as above. This is easy to verify by induction on j using the relations above.

We now turn to $\mathbf{h}$, for which the calculation is similar. From (15) we have

$$(i + b_j)h_{ij} = (i - 1 + \delta_{in})h_{i-1,j} + \frac{c_2(j - 1 + \delta_{out})}{c_1} h_{i,j-1} + \frac{\gamma}{c_1}\mathbf{1}_{\{i=1,j=0\}}. \tag{22}$$

Again skipping much of the algebra, for $j = 0$ we see that $h_{00} = 0$, while

$$h_{i0} = \gamma \frac{b_0!}{c_1 \delta_{in}!} \frac{(i - 1 + \delta_{in})!}{(i + b_0)!}$$

for all $i \geq 1$.

If $\gamma\delta_{out} = 0$, then $h_{ij} = 0$ is zero for all $j > 0$, so let us now assume $\gamma\delta_{out} > 0$. This time the boundary condition implies that $h_{0j} = 0$ for all j. For $j \geq 1$ we thus have from (22) that

$$h_{ij} = \sum_{k=1}^{i} \frac{c_2(j - 1 + \delta_{out})}{c_1} h_{k,j-1} \frac{(i - 1 + \delta_{in})_{i-k}}{(i + b_j)_{i-k+1}}.$$

(The only difference from (17) is that the sum starts with $k = 1$.) Arguing as before, using the $s = 1$ case of (20), we see that, for $i \geq 1$ and $j \geq 0$,

$$h_{ij} = \sum_{r=0}^{j} B_{jr} \frac{(i - 1 + \delta_{in})!}{(i + b_r)!},$$

where

$$B_{jr} = (-1)^r \gamma \frac{(j - 1 + \delta_{out})!}{j!(\delta_{out} - 1)!} \binom{j}{r} \frac{b_r!}{c_1 \delta_{in}!}.$$

(This makes sense as we are assuming that $\delta_{out} > 0$.) Here the $r = 0$ term dominates, and we see that for each $j \geq 0$ we have

$$h_{ij} \sim C_j'' i^{-1+\delta_{in}-b_0} = C_j'' i^{-(1+1/c_1+c_2\delta_{out}/c_1)} \tag{23}$$

as $i \to \infty$, for some positive constant C_j''. Returning now to $\mathbf{f} = \mathbf{g} + \mathbf{h}$, considering $j \geq 1$ fixed and $i \to \infty$ we see that when $\gamma\delta_{out} > 0$, the contribution from $\mathbf{h}$ dominates, while if $\gamma\delta_{out} = 0$, this contribution is zero. Thus combining (21) and (23) proves (11).

The second part of Theorem 2, the proof of (12), follows by interchanging in- and out-degrees, α and γ and δ_{in} and δ_{out}.