Popularity based random graph models leading to a scale-free degree sequence

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Abstract

As an extremely simplified model for the growth of the world wide web, Dorogovtsev, Mendes and Samukhin [12] and Drinea, Enachescu and Mitzenmacher [13] introduced the following random graph model, which generalizes an earlier model of Barabási and Albert [2]: at each time step we add a new vertex incident to r edges. The other endpoints of these edges are chosen with probability proportional to their in-degrees plus an initial attractiveness ar, where a is a constant. For all a, $r \in \mathbb{N}$, we determine the asymptotic form of the degree distribution for most of the vertices. Confirming non-rigorous arguments in [12, 13], this shows that for such a, the proportion $P(d)$ of vertices of degree d almost surely obeys a power law, where $P(d)$ is of the form d^{-2-a} for large d. The case $a = 1$ (which corresponds to the model in [2]) was proved earlier by Bollobás, Riordan, Spencer and Tusnády [9].

1 Introduction

Although a great deal of research has been done on random graphs, only very recently has data on large real world networks become available. For instance, the world wide web may be viewed as a graph by considering the nodes or vertices to be the individual web-pages, while the connections or edges are links going from one document to another. Empirical studies of the world wide web and other large networks indicate structural characteristics which differ substantially from those of the graph models used in most previous research.

In particular, experimental evidence collected by several authors (see for example [4] and [10]) suggests that the fraction $P_{in}(d)$ of web pages with a particular number d of in-links decreases as a scale-free power law, namely proportional to $d^{-\gamma_{in}}$, with $\gamma_{in} \simeq 2.1$. This is also the case for networks as diverse as the movie actor collaboration graph ($\gamma \simeq 2.3$, [3]), scientific collaboration networks ($\gamma \simeq 3$, [3]) and the Western US Power Grid ($\gamma \simeq 4$, [3]). On the other hand, in the most common random graph model $\mathcal{G}_{n,p}$ – where edges are included independently at random with probability p, and which was first investigated systematically by Erdős and Rényi (see e.g. [14]) – the fraction of

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vertices with a particular degree decreases exponentially as the distance from the mean increases. Another marked difference between, say the web, and $\mathcal{G}_{n,p}$ is that the links of the web are statistically correlated whereas the edges of $\mathcal{G}_{n,p}$ are independent.

Several new types of models have been introduced and studied recently in an effort to explain these observed phenomena. The simplest example of a model based on popularity (which was proposed by Barabási and Albert in [2], and with Jeong in [3]) is the following. At each time step add a new vertex v and a fixed number r of edges originating at v and directed towards vertices chosen at random with probability proportional to their degrees. Based on simulation and a heuristic continuous approximation, the above authors predicted that $P_{in}(d)$ behaves like d^{-3} for all $r \geq 1$. Using a combinatorial approach, this was then confirmed by Bollobás, Riordan, Spencer and Tusnády [9]. In order to generate power laws with arbitrary exponents, Dorogovtsev, Mendes and Samukhin [12] and Drinea, Enachescu and Mitzenmacher [13] introduced the following natural generalization of the above model: the destination of the r new edges added at each time step are chosen with probability proportional to the in-degrees plus an initial attractiveness ar. They gave a non-rigorous argument that $P_{in}(d)$ behaves like d^{-2-a}, for large d. Based on the approach of [9], we shall prove that this is correct for all $a \in \mathbb{N}$ and $r \in \mathbb{N}$.

Using different methods, Cooper and Frieze [11] recently studied (amongst others) a model where a new vertex picks its neighbours uniformly at random from the existing vertices with some fixed probability β and with the remaining probability picks them with probability proportional to the current vertex degrees. It is easily seen that this is asymptotically equivalent to the above model (with $\beta = (a-1)/(a+1)$). Cooper and Frieze obtained the order of magnitude of $P_{in}(d)$ for any $\beta < 1$, which corresponds to all $a \in \mathbb{R}$ with $a \geq 1$. However, our results for $a \in \mathbb{N}$ are considerably more precise and give additional insight into the evolution of the process. In particular, in Theorem 2 we obtain a very natural asymptotic expression for the degree distribution of most vertices. Furthermore, in Theorem 1, we obtain an asymptotic formula for $P_{in}(d)$ instead of only the order of magnitude and allow d to be a small polynomial in the total number of vertices instead of a logarithmic function.

Further random graph models for the web which also lead to a scale-free degree distribution were proposed by Kumar et al. [17]. Here for each new vertex v we choose a "prototype" vertex w uniformly at random from among the existing vertices. For $1 \leq i \leq r$, the ith out-neighbour of v is then chosen to be equal to the ith out-neighbour of w with probability $1 - \alpha$ and chosen uniformly at random from the whole set of existing vertices with probability α. For $r = 1$ this is easily seen to be asymptotically equivalent to the model considered in this paper with $\alpha = a/(a+1)$ (and for $r > 1$ should lead to a similar degree distribution). The degree distribution for $r = 1$ was investigated by [17]. While in this case they obtain the same order of magnitude for $P_{in}(d)$ as we do, for $d > 0$ the constants which they claim are not in agreement with those in Theorem 1 (and thus also not with those in Bollobás et al. [9]).

2 The model and results

Bollobás and Riordan [8] gave a mathematically precise definition of the process introduced by Barabási and Albert [2]. This definition easily extends to the model of Dorogovtsev et al. [12] and Drinea et al. [13]. First, we consider the case $r = 1$. G_0^1 is the empty graph, G_1^1 is the graph with the vertex 1 and one loop. Then form G_n^1 from G_{n-1}^1 by adding the vertex n and one directed edge from n to a vertex $i \in V_{n-1}$ where i is chosen randomly with

$$\mathbb{P}(i = j) = \begin{cases} \frac{d^{in}_{G^1_{n-1}}(j)+a}{(a+1)n-1} & \text{if } 1 \leq j \leq n-1 \\ \frac{a}{(a+1)n-1} & \text{if } j = n. \end{cases} \tag{1}$$

In other words, a vertex receives a link from a new vertex with probability proportional to its popularity, measured by the in-degree $d^{in}_{G^1_{n-1}}(j)$ plus an initial attractiveness a. We can extend this definition to a model where each vertex has constant out-degree $r > 1$. First run the graph process $\left(G_n^1\right)_{n \geq 0}$ as above and then contract consecutive sets of r vertices together to form the vertices of G_n^r. Suppressing the dependence on a, we shall write $\mathcal{G}_n^r$ for the probability space of directed graphs on n vertices $\{1, 2, \ldots, n\}$ where a random $G_n^r \in \mathcal{G}_n^r$ has the distribution derived from the process described above. Note that we can also form G_n^r from G_{n-1}^r by adding the r new edges one by one. The probability that the k^{th} new edge connects to vertex j is then proportional to the current indegree of j plus ra unless $j = n$, in which case the additive constant ar is replaced by ak.

Let $N_n^r(d)$ be the number of vertices with in-degree d in G_n^r. Our main result is that for $a \in \mathbb{N}$ the degree sequence is scale-free in the sense that the asymptotic form of the degree distribution is independent of n, the "scale" of the network.

Theorem 1. *Let $r \geq 1$ and $a \geq 1$ be fixed natural numbers, and let $(G_n^r)_{n \geq 0}$ be the graph process defined above. Let $\varepsilon > 0$ and define*

$$P_{in}(d) := (a+1)(ar+a)! \binom{d+ar-1}{ar-1} \frac{d!}{(d+ar+a+1)!}.$$

Then with probability tending to 1 as $n \to \infty$ we have

$$(1-\varepsilon)P_{in}(d) \leq \frac{N_n^r(d)}{n} \leq (1+\varepsilon)P_{in}(d)$$

for all d in the range $0 \leq d \leq n^{1/100(a+1)}$. In particular, for all d in this range, with probability tending to 1 we have

$$\frac{N_n^r(d)}{n} = \Theta\left(d^{-2-a}\right).$$

As mentioned in the introduction, the case $a = 1$ of this result is due to Bollobás et al. [9]. (Improving on the estimates in [3], Krapivsky, Redner and Leyvraz [16] had also predicted the precise result for the $a = 1$ case using non-rigorous arguments). We remark that the proof in [9] (and ours) uses an alternative description of the above process in terms of pairings (described in the following section), which was introduced by Bollobás and Riordan [8] in order to determine the likely diameter of a graph produced by the above process.

The case $a = 1$ and $r = 1$ of the above model is very similar to an existing model for random nonuniform trees. Szymański [20] proved that in this model, the expected proportion of vertices of degree d is $P_{in}(d)$, where $P_{in}(d)$ is defined as above.

Dorogovtsev, Mendes and Samukhin [12] gave a non-rigorous argument (which moreover only makes sense for fixed d) that in fact for all $a \in \mathbb{R}$ with $a > 0$ and all r, $d \in \mathbb{N}$,

$$\frac{1}{n}\mathbb{E}[N_n^r(d)] = (1+o(1))(a+1)\frac{\Gamma[a(r+1)+1]\,\Gamma[d+ar]}{\Gamma[ar]\,\Gamma[d+ar+a+2]}.$$

Noting that $\Gamma[x] = (x-1)!$ for $x \in \mathbb{N}$, Theorem 1 shows that for $a \in \mathbb{N}$ their claim is indeed correct.

Our second main result (from which Theorem 1 will follow by summing over the vertices) yields the degree distribution of all vertices which were not introduced to early or to late in the process.

Theorem 2. *For any a, $r \in \mathbb{N}$ and any k with $n^{1-1/20(a+1)} \leq k \leq n - n^{1-1/20(a+1)}$ and $0 < d \leq n^{1/100(a+1)}$, the in-degree d_k^{in} of the k^{th} vertex satisfies*

$$\mathbb{P}(d_k^{in} = d) = o(n^{-1}) + (1+o(1))\binom{d+ar-1}{ar-1} p^{ra}\,(1-p)^d,$$

where $p := (k/n)^{1/(a+1)}$.

In other words, the probability that a vertex k has indegree d is approximately the same as that the ar^{th} tail in a sequence of coin tosses appears in the $(d+ar)^{th}$ coin toss, where the probability of a tail is p (i.e. the distribution is approximately negative binomial). An explicit expression for $\mathbb{E}[d_k^{in}]$ is stated in Corollary 7.

3 Matchings and patterns

In much of the following analysis we shall make use of an alternative description of G_n^1. This is a generalization of the pairing model, for the case $a = 1$, which was introduced by Bollobás and Riordan [8]. Roughly speaking, an $(a,1)$-*matching* M_n^a is a particular type of partition of the set $[(a+1)n] := \{1, 2, \ldots, (a+1)n\}$ into n sets each containing $a+1$ elements, where within each set a elements are associated with the remaining element. More precisely, we define an $(a,1)$-matching $M_n^a \in \mathcal{M}_n^a$ inductively as follows. We view the natural numbers as points on the x-axis. M_1^a consists of the first $a+1$ natural numbers $[a+1]$, where a of them are joined, by directed chords, to the remaining one, which is not equal to $a+1$. Given an $(a,1)$-matching $M_{n-1}^a \in \mathcal{M}_{n-1}^a$ form M_n^a by adding $a+1$ points as follows: first append a points at the end and then choose the position of the remaining point uniformly at random from among all available $(a+1)n-1$ spaces. Here a space occurs before each of the $(a+1)(n-1)$ points in M_{n-1}^a and before each of the a new points at the end. After each inductive step, we renumber the points from left to right. See the top diagram in Fig. 1. With $a = 1$, M_n^1 is a pairing on $[2n]$. The latter structures are sometimes referred to as linearized chord diagrams in the literature, see for example [7]. Note that there are $\prod_{i=1}^{n}((a+1)i-1)$ many $(a,1)$-matchings in $\mathcal{M}_n^a$.

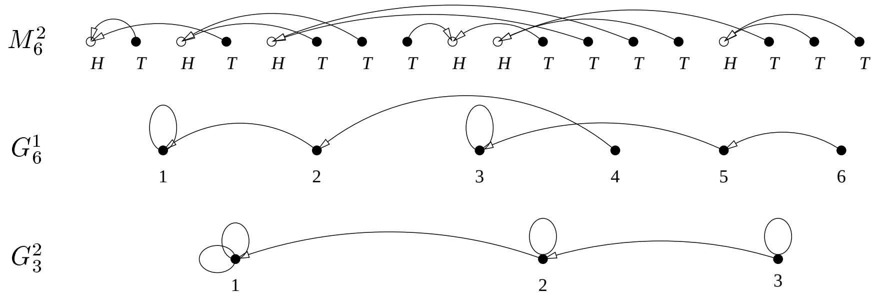

Figure 1: The top diagram represents a particular instance of an $M_6^2 \in \mathcal{M}_6^2$ on the set $[18] = [(a+1)n]$ and the associated (H,T)-pattern. Beneath is its transformation $\phi(M_6^2) = G_6^1 \in \mathcal{G}_6^1$ and then finally the merging of consecutive sets of $r = 2$ vertices in G_6^1 to form a G_3^2.

Later on, it will be useful to consider the $(a+1)n$ points as labeled with n Hs and an Ts, so that the head of a directed chord hits an H while the tails originate at the Ts. Such a labeling will be referred to as an *(H,T)-pattern*. Let $\mathcal{P}_n^a$ denote the set of all (H,T)-patterns which correspond to $(a,1)$-matchings in $\mathcal{M}_n^a$. Given some (H,T)-pattern $P \in \mathcal{P}_n^a$, we can calculate the number of corresponding $(a,1)$-matchings as follows. We assign integers from $[n]$ to consecutive blocks of a tails in P working from left to right. Each such block of tails must be connected to at least one head "to its left" (note this potentially includes heads which lie within the block of tails being considered). For $i = 1, 2, \ldots, n$, let $h(0,i;P)$ be the number of heads which could be connected to the i^{th} block of a tails in P. After each selection of a head, there will be one fewer available to the blocks of tails further to the right. Therefore the (H,T)-pattern P corresponds to exactly $\prod_{i=1}^{n}(h(0,i;P)-i+1)$ many $(a,1)$-matchings.

We construct a directed graph $\phi(M_n^a)$ on n vertices $\{v_1, v_2, \ldots, v_n\}$ from an $(a,1)$-matching M_n^a by proceeding from left to right and merging all points up to and including the a^{th} T to form the vertex v_1. The a chords originating at the a Ts form one edge in $\phi(M_n^a)$. Thus the vertex v_1 has in-degree $d^{in}(v_1)$ equal to the number of heads encountered so far and $d^{out}(v_1) = 1$. See Fig. 1. Then merge the following points up to and including the $(2a)^{th}$ T to form v_2, and so on. In general, $d^{in}(v_i)$ is equal to the number of heads encountered between the $(a(i-1))^{th}$ and the $(ai)^{th}$ tails while the out-degree is always one. The following proposition is easily proved by induction on n, and so we omit the proof (the $a = 1$ case is proved in [9]). It implies that instead of $\mathcal{G}_n^1$, we may work with the uniform distribution on $\mathcal{M}_n^a$.

Proposition 3. *If M_n^a is chosen uniformly at random from the set of all* $(a, 1)$*-matchings $\mathcal{M}_n^a$, then $\phi(M_n^a)$ has the same distribution as a random $G_n^1 \in \mathcal{G}_n^1$.*

On two occasions we will make use of the inequality in the following lemma due to Azuma (1967) and Hoeffding (1963), see e.g. [14] p. 37. Before stating the lemma recall that a sequence of random variables $(X_t)_{t=0}^n$ is said to be a *martingale* if for all t

$$E[X_t \mid X_0, X_1, \dots, X_{t-1}] = X_{t-1}.$$

Lemma 4. *Let $(X_t)_{t=0}^n$ be a martingale with $|X_t - X_{t-1}| \le c_t$ for $t = 1, 2, \dots, n$. Then for every $x > 0$,*

$$\mathbb{P}\left(|X_n - X_0| \ge x\right) \le \exp\left(-\frac{x^2}{2\sum_{t=1}^n c_t^2}\right).$$

The kinds of martingale sequences that we shall encounter are often referred to as Doob martingales and are defined in terms of expected values. Let f be any real-valued function defined over the space $\mathcal{G}_n^r$. An example we shall use later is $N_n^r(d)$, the number of vertices with in-degree d in $G_n^r \in \mathcal{G}_n^r$. A standard theorem (see for example [19], p. 90) implies that, for $t = 0, 1, 2, \dots, n$, the sequence of random variables defined by $X_t := \mathbb{E}[f(G_n^r) \mid G_t^r]$ is a martingale. From the definition of X_t, we see that $X_0 = \mathbb{E}[f(G_n^r)]$ and $X_n = f(G_n^r)$. We shall make use of this fact in the proofs of Proposition 8 and our main result, Theorem 1. In particular, the lemma implies that rather than having to study the number $N_n^r(d)$ itself, it will be sufficient to examine $\mathbb{E}[N_n^r(d)]$.

4 Summary of the proof

The following outline of the proof is intended as a road map for the reader. Steps 2 and 3 form the core of the proof; and it is here that the method differs most from the proof of the special case $a = 1$ given by Bollobás et al. [9].

1. To begin with, the asymptotic behaviour of the expected degrees and their partial sums is studied. The partial sum of the in-degrees $h(k, k')$ of the vertices $\{k+1, k+2, \dots, k'\}$ in G_n^1 is defined as $h(k, k') := \sum_{j=k+1}^{k'} d_{G_n^1}^{in}(j)$. Then an application of the Azuma-Hoeffding inequality of Lemma 4 shows that $h(k, k')$ is concentrated about its expected value $\mathbb{E}[h(k, k')]$ with high probability (see Corollary 9), provided that k' is sufficiently large compared to k.

2. By an interval $(c, d]$ in an $(a, 1)$-matching or an (H, T)-pattern denote the set of points lying between the $(ac)^{th}$ and the $(ad)^{th}$ tail. We now consider some pattern P which has the property that $h(k, k')$ is close to $\mathbb{E}[h(k, k')]$ and consider the space of all $(a, 1)$-matchings whose pattern agrees with P outside $(k, k']$. In this space, define $\mathcal{A}_{k^*}$ to be the event that the number $h(k, k^*)$ of heads does not deviate significantly from its expected value, where k^* is somewhat smaller than k'. By considering an auxiliary probability space on $(k, k']$ where heads occur with probability

proportional to $h(k,k')$, we can use our newly acquired knowledge about $h(k,k')$ from Step 1 to show that the event $\mathcal{A}_{k^*}$ occurs with very high probability (see Lemma 13).

3. We now condition on $h(k,k')$ being close to $\mathbb{E}[h(k,k')]$ and on the event $\mathcal{A}_{k^*}$ and partition the space of all remaining matchings according to their patterns outside the interval $(k,k^*]$. In each subspace we can prove that choosing (H,T)-patterns uniformly at random in $(k,k^*]$ approximates the distribution of the corresponding $(a,1)$-matchings (see Lemma 14). This argument would fail if we tried to prove it directly for the larger interval $(k,k']$ using the same method.

4. In Section 7, using the law of total probability and the results proven about $h(k,k')$ and $\mathcal{A}_{k^*}$, we calculate $\mathbb{P}(d_k = d+1)$, the probability that the k^{th} vertex has in-degree d. In turn, this gives the expected number of vertices in G_n^1 with in-degree d as $\mathbb{E}[N_n^1(d)] = \sum_{k=1}^{n} \mathbb{P}(d_k = d+1)$. We then indicate how to modify the proof to the case $r > 1$, thus obtaining the required expression for $\mathbb{E}[N_n^r(d)]$. Finally, an application of the Azuma-Hoeffding inequality proves that $N_n^r(d)$ is sharply concentrated about its mean which gives us Theorem 1.

Throughout this paper, we will always assume that n is sufficiently large to satisfy our estimates.

5 Degrees and their sums

Working directly with the graph process $(G_n^1)_{n\geq 0}$, we may extract a certain amount of information about the vertex degrees. Let $D_{k,n}$ denote the sum of the degrees of the first k vertices in G_n^1. It is easily seen that given $D_{k,n-1}$, the probability that $D_{k,n} = D_{k,n-1}+1$ is equal to $p := \frac{D_{k,n-1}+k(a-1)}{(a+1)n-1}$. Thus

$$\begin{aligned}
\mathbb{E}[D_{k,n} \mid D_{k,n-1}] &= p(D_{k,n-1}+1) + (1-p)D_{k,n-1} \\
&= p + D_{k,n-1} \\
&= \frac{k(a-1)}{(a+1)n-1} + D_{k,n-1}\frac{(a+1)n}{(a+1)n-1}.
\end{aligned}$$

Taking the expected value of both sides gives

$$\mathbb{E}[D_{k,n}] = \frac{(a-1)k}{(a+1)n-1} + \mathbb{E}[D_{k,n-1}]\frac{(a+1)n}{(a+1)n-1}.$$

Now winding this recursion relation down as far as the boundary condition $\mathbb{E}[D_{k,k}] = 2k$ we obtain the following expression for $\mathbb{E}[D_k] := \mathbb{E}[D_{k,n}]$, namely

$$\frac{(a-1)k}{(a+1)n-1} + \sum_{\ell=k+1}^{n-1} \frac{(a-1)k}{(a+1)\ell-1} \prod_{i=\ell+1}^{n} \frac{(a+1)i}{(a+1)i-1} + 2k \prod_{i=k+1}^{n} \frac{(a+1)i}{(a+1)i-1}. \tag{2}$$

For the case $a = 1$, Bollobás et al. [9], using a different approach to the one we will take, showed indirectly that D_k has expected value about $2\sqrt{nk}$. We shall prove the following lemma which includes this result as a special case.

Lemma 5.

$$\mathbb{E}[D_k] = (1 + O(k^{-1/4}))\left((a+1)n^{\frac{1}{a+1}}k^{\frac{a}{a+1}} - (a-1)k\right),$$

for all k in the range $R \le k \le n - R$, where $R = n^{1-1/20(a+1)}$.

Proof. To begin with, we obtain upper and lower bounds for the products in the expression (2). To do so we shall make use of the inequalities $e^{\frac{x}{1+x}} < 1+x < e^x$ for $x > 0$ (see for example [1]) in each factor of the products and then bound the resulting sums using integrals. For the upper bound, first observe that

$$\begin{aligned}
\prod_{i=\ell+1}^{n} \frac{(a+1)i}{(a+1)i-1} &\le \exp\left(\sum_{i=\ell+1}^{n} \frac{1}{(a+1)i-1}\right) \\
&\le \exp\left(\left(1 + \frac{1}{(a+1)\ell}\right)\frac{1}{a+1}\sum_{i=\ell+1}^{n}\frac{1}{i}\right) \\
&\le \exp\left(\left(1 + \frac{1}{(a+1)\ell}\right)\frac{1}{a+1}\int_{\ell}^{n}\frac{1}{i}di\right) \\
&= \left(\frac{n}{\ell}\right)^{\frac{1}{(a+1)^2\ell}}\left(\frac{n}{\ell}\right)^{\frac{1}{a+1}} \qquad (3) \\
&= (1 + O(\ell^{-1}\log\left(\frac{n}{l}\right)))\left(\frac{n}{\ell}\right)^{\frac{1}{a+1}} \\
&= (1 + O(\ell^{-1/4}))\left(\frac{n}{\ell}\right)^{\frac{1}{a+1}}. \qquad (4)
\end{aligned}$$

For the second last line, we made use of the following relationships: $e^y = e^{\frac{x}{1+x}} < 1 + x = 1 + \frac{y}{1-y} \le 1 + 2y$, with $\frac{x}{1+x} := y := \ell^{-1}\log\frac{n}{\ell}$, for $0 < x, y < 1/2$. The lower bound proceeds in a similar manner, but now we need to translate the interval of summation/integration, or equivalently translate the variable of integration, to obtain the required result.

$$\begin{aligned}
\prod_{i=\ell+1}^{n} \frac{(a+1)i}{(a+1)i-1} &\ge \exp\left(\frac{1}{a+1}\sum_{i=\ell+1}^{n}\frac{1}{i}\right) \ge \exp\left(\frac{1}{a+1}\int_{\ell+1}^{n}\frac{1}{i+1}di\right) \\
&= \left(\frac{1+\frac{1}{n}}{1+\frac{2}{\ell}}\right)^{\frac{1}{(a+1)}}\left(\frac{n}{\ell}\right)^{\frac{1}{a+1}} \qquad (5) \\
&= (1 + O(\ell^{-1}))\left(\frac{n}{\ell}\right)^{\frac{1}{a+1}}.
\end{aligned}$$

Thus the third term in (2) may be written as

$$2k\prod_{i=k+1}^{n}\frac{(a+1)i}{(a+1)i-1} = (1 + O(k^{-1/4}))2n^{\frac{1}{a+1}}k^{\frac{a}{a+1}}.$$

Using (3), the second term in (2) may be bounded as follows.

$$
\begin{aligned}
&\sum_{\ell=k+1}^{n-1} \frac{(a-1)k}{(a+1)\ell-1} \prod_{i=\ell+1}^{n} \frac{(a+1)i}{(a+1)i-1} \\
&\leq \frac{(a-1)k}{a+1}\left(1+\frac{1}{(a+1)k-1}\right) \sum_{\ell=k+1}^{n} \frac{1}{\ell}\left(\frac{n}{\ell}\right)^{\frac{1}{(a+1)^2\ell}}\left(\frac{n}{\ell}\right)^{\frac{1}{a+1}} \\
&\leq \frac{(a-1)k}{a+1}(1+O(k^{-1/4}))\left(\frac{n}{k}\right)^{\frac{1}{(a+1)^2k}} n^{\frac{1}{a+1}} \sum_{\ell=k+1}^{n} \ell^{-1-\frac{1}{a+1}} \\
&\leq \frac{(a-1)k}{a+1}(1+O(k^{-1/4}))n^{\frac{1}{a+1}} \int_k^n \ell^{-1-\frac{1}{a+1}} d\ell \\
&= (1+O(k^{-1/4}))(a-1)k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}}-1\right).
\end{aligned}
$$

The lower bound is established in a similar way. In particular, using (5)

$$
\begin{aligned}
\sum_{\ell=k+1}^{n-1} \frac{(a-1)k}{(a+1)\ell-1} \prod_{i=\ell+1}^{n} \frac{(a+1)i}{(a+1)i-1}
&\geq (a-1)k\left(\frac{1+\frac{1}{n}}{1+\frac{1}{k}}\right)^{\frac{1}{a+1}}\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}}-1\right) \\
&= (1+O(k^{-1}))(a-1)k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}}-1\right).
\end{aligned}
$$

Thus taking all three terms of (2) into account we obtain

$$
\mathbb{E}[D_k] = O\left(\frac{k}{n}\right) + (1+O(k^{-1/4}))\left((a+1)n^{\frac{1}{a+1}}k^{\frac{a}{a+1}}-(a-1)k\right),
$$

which for the range of k under consideration gives us the required result. □

We remark that the result holds for a far larger range of vertices k than stated in Lemma 5, but throughout the proof we will only be interested in such vertices which are bounded well away from the extremes.

In our analysis we will often be interested in the sums of the in-degrees over particular "intervals" of vertices. Let $h(k,k') := \sum_{j=k+1}^{k'} d_n^{in}(j)$ be the sum of the in-degrees of the vertices $j \in \{k+1, k+2, \ldots, k'\}$. The notation $h(k,k')$ was chosen to suggest that the sum of the in-degrees corresponds to the number of heads in an "interval" of an $(a,1)$-matching. We will deal with this quantity in more detail in the next section. First we derive the following consequence of Lemma 5.

Corollary 6. *Let $\tilde{k} = (1+\tilde{\varepsilon})k$, where $0 < \tilde{\varepsilon} = o(n^{-1/3})$ then*

$$
\mathbb{E}[h(k,\tilde{k})] = (1+O(k^{-1/4}))a\tilde{\varepsilon}k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}}-1\right),
$$

where $R \leq k < \tilde{k} \leq n-R$, with $R = n^{1-\frac{1}{20(a+1)}}$.

Proof. Since D_k is the sum of the in-degrees of the first k vertices plus k, i.e. the sum of the out-degrees, we can calculate $\mathbb{E}[h(k,\tilde{k})]$ as follows.

$$\begin{aligned}\mathbb{E}[h(k,\tilde{k})] &= \mathbb{E}[D_{\tilde{k}}] - \tilde{k} - (\mathbb{E}[D_k] - k) \\ &= (1+O(k^{-1/4}))\left((a+1)n^{\frac{1}{a+1}}(\tilde{k}^{\frac{a}{a+1}} - k^{\frac{a}{a+1}}) - a(\tilde{k}-k)\right) \\ &= (1+O(k^{-1/4}))\left((a+1)n^{\frac{1}{a+1}}k^{\frac{a}{a+1}}((1+\tilde{\varepsilon})^{\frac{a}{a+1}} - 1) - a(\tilde{k}-k)\right)\end{aligned}$$

and now using the binomial expansion to express $(1+\tilde{\varepsilon})^{\frac{a}{a+1}}$ as $1+\frac{a\tilde{\varepsilon}}{a+1} + O(\tilde{\varepsilon}^2)$ for $0 < \tilde{\varepsilon} = o(1)$ we obtain

$$\mathbb{E}[h(k,\tilde{k})] = (1+O(k^{-1/4}))a\tilde{\varepsilon}k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}} - 1\right) + O(\tilde{\varepsilon}^2 n^{\frac{1}{a+1}}k^{\frac{a}{a+1}}),$$

from which the result follows, since the assumption on the range of k and $\tilde{\varepsilon}$ implies that $k^{-1/4}((\frac{n}{k})^{1/(a+1)} - 1)$ is sufficiently large compared to $\tilde{\varepsilon}\left(\frac{n}{k}\right)^{1/(a+1)}$. □

Setting $\tilde{\varepsilon}k$ equal to r gives the following corollary. We will not need it in the proof of Theorems 1 or 2 (note that it follows directly from the latter) but state it separately here since we believe it is of independent interest.

Corollary 7. *Suppose that $R \le k \le n - R$, with $R = n^{1-\frac{1}{20(a+1)}}$. Then*

$$\mathbb{E}[d^{in}_{G^r_n}(k)] = (1+o(1))\, ar\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}} - 1\right).$$

We would like to show that in a random G^1_n, the probability that $h(k,k')$ deviates substantially from the expected in-degree sum $\mathbb{E}[h(k,k')]$ is negligible. In order to establish this result we shall first prove the following proposition, which relies on the Azuma-Hoeffding inequality (Lemma 4) and shows that the degree sum D_k is sharply concentrated about its mean.

Proposition 8. *Let $D_k = \sum_{j=1}^{k} d_{G^1_n}(j)$. Then, for all $k \ge R = n^{1-1/20(a+1)}$ in $G^1_n \in \mathcal{G}^1_n$ we have*

$$\mathbb{P}\left(|D_k - \mathbb{E}[D_k]| \ge 3\sqrt{n}\log n\right) = o(n^{-1}).$$

Proof. For $t \ge k$, let Y_t be defined as $\mathbb{E}[D_k \mid G^1_t]$. Then $Y_k = \mathbb{E}[D_k]$ and $Y_n = D_k$. As we mentioned earlier $(Y_t)^n_{t=k}$ is a Doob martingale. We wish to use the Azuma-Hoeffding inequality (Lemma 4) so we must show that the differences $|\,Y_t - Y_{t-1}|$ are bounded for all $t = k+1,\ldots,n$. We shall write $Y_t = \mathbb{E}[D_k \mid G^1_t]$ as a function $Y_t(I_{k+1},\ldots,I_t)$ of indicator random variables $I_{k+1},\ldots,I_t$, where $I_t = 1$ if the t^{th} vertex sends an edge to one of the first k vertices and $I_t = 0$ otherwise. In terms of (H,T)-patterns $I_t = 1$ implies that the $(ak)^{th}$ tail T has been moved one step to the right at time t. Let $J_{t-1} = (I_{k+1},\ldots,I_{t-1})$ correspond to some fixed choice of I_i for $i = k+1,\ldots,t-1$. Then from the definition of Y_t it follows immediately that, with $p := \mathbb{P}(I_t = 1 |\, J_{t-1})$, we have

$$Y_{t-1}(J_{t-1}) = p\, Y_t(J_{t-1}, I_t = 1) + (1-p)\, Y_t(J_{t-1}, I_t = 0).$$

Thus we see that

$$|\, Y_t(J_{t-1}, I_t = 1) - Y_{t-1}(J_{t-1})| \leq (1-p)\,|\, Y_t(J_{t-1}, I_t = 1) - Y_t(J_{t-1}, I_t = 0)|$$

while

$$|\, Y_t(J_{t-1}, I_t = 0) - Y_{t-1}(J_{t-1})| \leq p\,|\, Y_t(J_{t-1}, I_t = 1) - Y_t(J_{t-1}, I_t = 0)|$$

Therefore, instead of bounding $|\, Y_t - Y_{t-1}|$ it suffices to bound $|\, Y_t(J_{t-1}, I_t = 1) - Y_t(J_{t-1}, I_t = 0)|$. Thinking in terms of an (H,T)-pattern, let S denote the point, be it an H or a T, which came directly before the $(ak)^{th}$ T at time t, conditional on $\{J_{t-1}, I_t = 1\}$. Then one easily sees that the difference of the two expected values $Y_t(J_{t-1}, I_t = 1)$ and $Y_t(J_{t-1}, I_t = 0)$ is equal to the expected number B of points in G_n^1 which have been inserted between S and the $(ak)^{th}$ T (including S). Let B_{t+i} be the number of points (including S) between S and the $(ak)^{th}$ T at time $t+i$. Then calculating as we did above for the expected degree (note that the value of J_{t-1} is irrelevant here) we get

$$B = \mathbb{E}[B_n] = \frac{(a+1)n}{(a+1)n-1}\mathbb{E}[B_{n-1}]$$

and winding this recursion down as far as $\mathbb{E}[B_t] = 1$ gives the following expression for $\mathbb{E}[B_n]$.

$$\mathbb{E}[B_n] = \prod_{i=t+1}^{n} \frac{(a+1)i}{(a+1)i-1}.$$

Now since $t \geq k \geq R = n^{1-1/20(a+1)}$, we may use (4) to obtain

$$|\, Y_t - Y_{t-1}| \leq \mathbb{E}[B_n] \leq (1 + O(t^{-1/4}))\left(\frac{n}{t}\right)^{\frac{1}{a+1}} \leq 2\left(\frac{n}{t}\right)^{\frac{1}{2}} =: c_t$$

We are now in a position to apply the Azuma-Hoeffding inequality (Lemma 4) with

$$\sum_{t=k}^{n} {c_t}^2 = 2n \sum_{t=k}^{n} t^{-1} \leq 2n \log n$$

which gives us the required result. □

We shall make repeated use of the following concentration result for $h(k,k')$.

Corollary 9. *For $R \leq k \leq n - R$, where $R = n^{1-1/20(a+1)}$, let $k' = k + n^{1/2+1/5(a+1)}$ and define $\mathcal{A}_{k'}$ as the event $\{|D_\ell - \mathbb{E}[D_\ell]| \leq 3\sqrt{n}\log n : \ell = k, k'\}$ and let $h(k,k') = \sum_{j=k+1}^{k'} d_n^{in}(j)$. Then in $G_n^1 \in \mathcal{G}_n^1$, $\mathbb{P}(\overline{\mathcal{A}_{k'}}) = o(n^{-1})$. In particular,*

$$\mathbb{P}(\,|h(k,k') - \mathbb{E}[h(k,k')]|\,) \geq 6\sqrt{n}\log n) = o(n^{-1}).$$

Proof. The corollory is an immediate consequence of the following inequality.

$$\begin{aligned}|h(k,k') - \mathbb{E}[h(k,k')]| &= |D_{k'} - k' - D_k + k - \mathbb{E}[D_{k'} - k' - D_k + k]| \\ &\leq |D_{k'} - \mathbb{E}[D_{k'}]| + |D_k - \mathbb{E}[D_k]|.\end{aligned}$$

□

Bollobás et al. [9] in their analysis of the case $a = 1$ also proved a concentration result for the number of heads in terms of D_k although using quite different means. Due to the simpler pairing model of the graph process they were able to calculate the necessary probabilities by directly counting the appropriate events. Unfortunately, the same approach breaks down for $a > 1$. The counting problems become intractable due to the relative freedom that the a tails corresponding to a single vertex have to place themselves.

6 Heads and tails in intervals

Our overall objective is to calculate the expected number of vertices in G_n^1 with in-degree d. As an intermediate step, we shall first determine the probability that $d_k := d_{G_n^1}(k)$ is equal to $d+1$, i.e. that the in-degree of the k^{th} vertex is d, for a large range of k. In order to do this, it will be necessary to examine the relationship between the distribution of $(a,1)$-matchings and the distribution of (H,T)-patterns more closely. The main results of this section are Lemmas 13 and 14. Lemma 13 shows that we can restrict our attention to $(a,1)$-matchings where the number of heads in intervals of length slightly less than $\sqrt{n}$ is close to the expected number. Lemma 14 then establishes that, for these $(a,1)$-matchings, the probabilities of the events of interest can be calculated by counting the underlying (H,T)-patterns.

We shall define intervals in a pattern or matching as follows. For $0 \le k < \tilde{k} \le n$, let $(k,\tilde{k}]$ denote the set of points after the $(ak)^{th}$ tail up to and including the $(a\tilde{k})^{th}$ tail. It will also be convenient to use counter functions for tails and heads in an interval; let $t(k,\tilde{k})$ be the number of tails in $(k,\tilde{k}]$ and recall that $h(k,\tilde{k})$ is the number of heads in $(k,\tilde{k}]$, i.e. $h(k,\tilde{k}) = \sum_{j=k+1}^{\tilde{k}} d_n^{in}(j)$. Since the intervals are defined on the tails we see immediately that $t(k,\tilde{k}) = a(\tilde{k}-k)$. In order to avoid confusion, we shall often use the notation $h(k,\tilde{k};P)$ to indicate the particular (H,T)-pattern P being considered. Similarly $D_k(P)$ denotes the degree sum given a particular (H,T)-pattern P, i.e. $D_k(P) = h(0,k;P) - k$. Note that this corresponds with the notation used in Section 3: Recall that there we defined $h(0,i;P)$ as the number of heads which could be connected to the i^{th} block of a tails and that is precisely the number of heads in the interval $(0,i]$ in an (H,T)-pattern P.

In our analysis we shall be interested in two particular types of intervals. Throughout, we shall assume that k is an integer in the range $R \le k \le n-R$, with $R = n^{1-1/20(a+1)}$. The first type of interval, which we denote by $(k,k^*]$, has length defined by

$$k^* = k + n^{1/2-1/5(a+1)} = (1+\varepsilon^*)k, \tag{6}$$

while the second type of interval, denoted by $(k,k']$, has length defined by

$$k' = k + n^{1/2+1/5(a+1)} = (1+\varepsilon')k. \tag{7}$$

In order to prepare for the proof of Lemma 13, we shall first introduce two auxiliary probability spaces. Recall that $\mathcal{A}_{k'}$ was defined in Corollary 9 and we showed that if an $(a,1)$-matching or (H,T)-pattern satisfies $\mathcal{A}_{k'}$ then $\{|h(k,k')-E[h(k,k')]| \le 6\sqrt{n}\log n\}$ is also fulfilled. Suppose that we are given an (H,T)-pattern $P \in \mathcal{P}_n^a$ satisfying $\mathcal{A}_{k'}$. Let $M = h(k,k';P)$ and define Γ_M to be the set of all (H,T)-patterns which agree with P outside the interval $(k,k']$. Define $(\Gamma_M, \mathbb{P}_{\Gamma_M})$ as the probability space where the elements of Γ_M are chosen uniformly at random.

The second auxiliary space is related to the probability space Γ_M and has the advantage of being even more tractable. Suppose P and M are as defined above, then let $N = t(k,k') + h(k,k';P)$ and define $\Gamma_p = \Gamma_{M/N}$ to be the space of all (H,T)-patterns generated by N independent random trials, where in each trial the probability of a head occurring is $p = M/N$ and that of a tail is $1-p$. We shall denote the probability measure over this set by $\mathbb{P}_{\Gamma_p}$. This binomially distributed probability space will be referred to as $(\Gamma_p, \mathbb{P}_{\Gamma_p})$ or simply Γ_p. Just as for the random graph models $\mathcal{G}_{n,p}$ and $\mathcal{G}_{n,M}$, there is some overlap in the notation Γ_p and Γ_M, though it is always clear from the context what we are talking about.

We first need to establish bounds for $\frac{p}{1-p}$ and p, which we shall make use of on several occasions.

Proposition 10. *Let p, k and k' be as defined above. Then we have*

$$\Omega(n^{-\frac{1}{20(a+1)}}) = \frac{p}{1-p} = O(n^{\frac{1}{20(a+1)}}),$$

for all k in the range $R \le k \le n-R$, with $R = n^{1-1/20(a+1)}$. In particular, $p = \Omega(n^{-\frac{1}{20(a+1)}})$.

Proof. Using Corollary 6 with $k' = k + n^{1/2+1/5(a+1)} = (1+\varepsilon')k$ we can crudely bound $\frac{p}{1-p}$ from above as follows.

$$\begin{aligned} \frac{p}{1-p} = \frac{h(k,k')}{t(k,k')} &= \frac{\mathbb{E}[h(k,k')] + O(\sqrt{n}\log n)}{t(k,k')} \\ &= O\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}}\right) = O(n^{\frac{1}{20(a+1)}}). \end{aligned}$$

Now for the lower bound:

$$\begin{aligned} \frac{p}{1-p} &= \Omega\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}} - 1\right) + O(n^{-1/5(a+1)}\log n) \\ &= \Omega\left(\left(1 + \frac{n^{-1/20(a+1)}}{1-n^{-1/20(a+1)}}\right)^{\frac{1}{a+1}} - 1\right) + O(n^{-1/6(a+1)}) \\ &= \Omega\left(n^{-1/20(a+1)}\right). \end{aligned}$$

The lower bound for p follows immediately by solving the above lower bound on $p/(1-p)$ for p. $\square$

We will actually be interested in $h(k,k^*)$ but as a preliminary step we shall study the number of heads $h_p(k,k^*)$ which occur in Γ_p before $t(k,k^*) = a(k^*-k)$

tails have appeared. For appropriate values of x and δ, we shall denote by $[(1 \pm \delta)x]$ the set of natural numbers $i \in \mathbb{N}$ such that $(1-\delta)x \le i \le (1+\delta)x$. The following proposition provides a concentration result for $h_p(k, k^*)$.

Proposition 11. *Let $\mathcal{A}_{k^*,p}$ be the event $\{h_p(k,k^*) \in [(1\pm\delta)\frac{p}{1-p}t(k,k^*)]\}$, where $\delta = n^{-1/90(a+1)}$. Then*

$$\mathbb{P}_{\Gamma_p}(\overline{\mathcal{A}_{k^*,p}}) \le 2\exp(-n^{\frac{1}{3}}).$$

Proof. Label the individual trials in Γ_p with the natural numbers $\{1, 2, \ldots, N\}$ and let $t_p(x)$ be the number of tails in the first $x \le N$ trials. Likewise, let $h_p(x)$ be the number of heads in the first x trials. Our approach shall be to establish an upper and lower bound for $h_p(k, k^*)$. To do so, we shall apply the Chernoff inequality to the quantity $t_p(x)$.

Suppose that $(Y_i)_{i=1}^x$ are independent indicator random variables (e.g. Y_i takes the values 1 or 0, in our case H or T) such that for $1 \le i \le x$, $\mathbb{P}(Y_i = 1) = p$, where $0 < p < 1$. Then, for $Y = \sum_{i=1}^x Y_i$ and any $0 < \tau \le 3/2$, the Chernoff inequality (see for example [14], p. 27) holds:

$$\mathbb{P}\left(|\, Y - \mathbb{E}[Y]\,| \ge \tau\mathbb{E}[Y]\right) \le 2\exp\left(-\frac{\tau^2}{3}\mathbb{E}[Y]\right). \tag{8}$$

This is precisely the situation we have for the random variable $t_p(x)$ in Γ_p, where $1-p$ is the probability of getting a tail. Let $\tau = n^{-1/16(a+1)}$ and define τ_0 by $(1-\tau_0)(1+\tau) = 1$. Since $\mathbb{E}_{\Gamma_p}[t_p(x)] = (1-p)x$ and $\tau_0 = (1+o(1))\tau$, we obtain

$$\begin{aligned}\mathbb{P}_{\Gamma_p}\left(t_p\left((1+\tau)\frac{t(k,k^*)}{1-p}\right) \le t(k,k^*)\right) &\le 2\exp\left(-\frac{\tau_0^2}{3}(1+\tau)t(k,k^*)\right)\\ &< \exp(-n^{\frac{1}{3}})\end{aligned}$$

Thus with probability at least $1 - \exp(-n^{1/3})$, more than $t(k, k^*)$ tails are observed in the first $(1+\tau)\frac{t(k,k^*)}{1-p}$ trials. Since by Proposition 10 we have $\tau/p = O(n^{-1/80(a+1)})$, we may conclude that

$$\begin{aligned}h_p(k,k^*) &\le (1+\tau)\frac{t(k,k^*)}{1-p} - t(k,k^*) = (1+\frac{\tau}{p})\frac{p}{1-p}t(k,k^*)\\ &\le (1+\delta)\frac{p}{1-p}t(k,k^*),\end{aligned}$$

The proof of the lower bound for $h_p(k, k^*)$ is very similar. Applying the Chernoff inequality (8) we have

$$\begin{aligned}\mathbb{P}_{\Gamma_p}\left(t_p\left((1-\tau)\frac{t(k,k^*)}{1-p}\right) \ge t(k,k^*)\right) &\le 2\exp\left(-\frac{\tau_1^2}{3}(1-\tau)t(k,k^*)\right)\\ &< \exp(-n^{\frac{1}{3}}),\end{aligned}$$

where τ_1 is defined by $(1+\tau_1)(1-\tau) = 1$. Therefore with probability at least $1 - \exp(-n^{1/3})$, not enough tails (i.e. $t(k,k^*)$) will occur in order to complete the count of $h_p(k, k^*)$ and so we obtain

$$h_p(k,k^*) \ge (1-\tau)\frac{t(k,k^*)}{1-p} - t(k,k^*) \ge (1-\delta)\frac{p}{1-p}t(k,k^*),$$

which completes the proof. $\square$

Combined with the previous proposition, the following calculation shows that if $\mathcal{A}_{k^*,p}$ and $\mathcal{A}_{k'}$ are satisfied, then $h_p(k,k^*)$ is close to $\mathbb{E}[h(k,k^*)]$. In Lemma 13 we shall show that this is also true for $h(k,k^*)$.

$$\begin{aligned}\frac{p}{1-p}t(k,k^*) &= \frac{t(k,k^*)}{t(k,k')}h(k,k') \\ &= \frac{\varepsilon^*}{\varepsilon'}(\mathbb{E}[h(k,k')] + O(\sqrt{n}\log n)) \\ &= (1+O(\delta))a\varepsilon^* k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}} - 1\right). \end{aligned} \tag{9}$$

In the second line, we applied the assumption that $\mathcal{A}_{k'}$ holds and in the last line, we applied Corollary 6 and the fact that the assumption on the range of k implies that the error term $O(\sqrt{n}\log n))$ is negligible. (In the last line we left some room in order to write the expression in the form we will use in Section 7.)

Our next step is to use Proposition 11 to establish a similar result in the uniformly distributed probability space Γ_M. Let $h_M(k,k^*)$ denote the number of heads in the interval $(k,k^*]$ (i.e. between the $(ak)^{th}$ and $(ak^*)^{th}$ tails) in an (H,T)-pattern in Γ_M.

Corollary 12. *Let $\mathcal{A}_{k^*,M}$ be the event $\{h_M(k,k^*) \in [(1\pm\delta)\frac{p}{1-p}t(k,k^*)]\}$. Then*

$$\mathbb{P}_{\Gamma_M}(\overline{\mathcal{A}_{k^*,M}}) \leq 6(a+1)n\exp(-n^{\frac{1}{3}})$$

Proof. We may use the following inequality (10) due to Pittel ([14], p. 17) to transfer attention from Γ_p to the related uniformly distributed model Γ_M. If $\mathcal{Q}$ is any property which the elements of Γ_M and Γ_p may possess then

$$\mathbb{P}_{\Gamma_M}(\mathcal{Q}) \leq 3\sqrt{M}\mathbb{P}_{\Gamma_p}(\mathcal{Q}), \tag{10}$$

where we have labeled the probabilities to indicate the underlying probability spaces. The result now follows by noting that $\sqrt{M} = \sqrt{h(k,k')} \leq (a+1)n$ and that the events $\mathcal{A}_{k^*,p}$ and $\mathcal{A}_{k^*,M}$ imply the same property on subsets of (H,T)-patterns in Γ_p and Γ_M, respectively. $\square$

We are now ready to turn our attention to probability spaces of $(a,1)$-matchings. The first probability space is a natural extension of the space Γ_M of (H,T)-patterns on $(k,k']$ to include the corresponding $(a,1)$-matchings. Recall that Γ_M was defined in terms of a given (H,T)-pattern P satisfying $\mathcal{A}_{k'}$. Let $\mathcal{M}'(P)$ denote the set of $(a,1)$-matchings in $\mathcal{M}_n^a$ whose (H,T)-patterns agree with P outside $(k,k']$. Define the probability $\mathbb{P}_{\mathcal{M}'(P)}$ on $\mathcal{M}'(P)$ by choosing the elements uniformly at random.

Lemma 13. *Let $\mathcal{A}_{k^*}$ denote the event $\{h(k,k^*) \in [(1\pm\delta)\frac{p}{1-p}t(k,k^*)]\}$. Then*

$$\mathbb{P}_{\mathcal{M}'(P)}(\overline{\mathcal{A}_{k^*}}) = o(n^{-1}).$$

Proof. Since in $\mathcal{M}'(P)$ we are considering matchings, while the elements of Γ_M are patterns, we need to account for the weighting. Given two (H,T)-patterns P_1 and P_2 which are identical with P outside $(k,k']$, for $i=1,2$, let $\mathcal{M}_i$ denote the set of all $(a,1)$-matchings which have (H,T)-pattern P_i. Note that $\mathcal{M}_i \subseteq \mathcal{M}'(P)$.

As was pointed out in Section 3, the number of $(a,1)$-matchings on a particular (H,T)-pattern P is given by the product $\prod_{i=1}^{n}(h(0,i;P)-i+1)$ where $h(0,i;P)$ is the number of heads which could be connected to the i^{th} block of a tails, where the blocks of tails are numbered from left to right. Thus since P_1 and P_2 agree with the (H,T)-pattern P outside $(k,k']$, it is easily seen that

$$
\begin{aligned}
\frac{|\mathcal{M}_1|}{|\mathcal{M}_2|} &= \frac{\prod_{i=1}^{n}(h(0,i;P_1)-i+1)}{\prod_{i=1}^{n}(h(0,i;P_2)-i+1)} \\
&\leq \prod_{i=k+1}^{k'} \frac{h(0,k';P_1)-k+1}{h(0,k;P_2)-k'+1} \\
&= \prod_{i=k+1}^{k'} \frac{D_{k'}(P_1)-k'-k+1}{D_k(P_2)-k-k'+1} \\
&= \left(1+\frac{D_{k'}(P)-D_k(P)}{D_k(P)-k-k'+1}\right)^{k'-k} \\
&\leq \exp\left((k'-k)\frac{h(k,k';P)+k'-k}{D_k(P)-k-k'+1}\right).
\end{aligned}
$$

We now give an upper bound for $h(k,k';P)+k'-k$. Since P satisfies $\mathcal{A}_{k'}$, recalling our assumption (7) on k and k', Corollary 6 implies

$$
\begin{aligned}
h(k,k')+k'-k &= O\left(\varepsilon' k\left(\frac{n}{k}\right)^{\frac{1}{a+1}}\right)+O(\varepsilon' k)+O\left(\sqrt{n}\log n\right) \\
&= O\left(n^{\frac{1}{2}+\frac{1}{5(a+1)}}\left(\frac{n}{k}\right)^{\frac{1}{a+1}}\right)+O(n^{\frac{1}{2}+\frac{1}{5(a+1)}}) \\
&= O\left(n^{\frac{1}{2}+\frac{1}{5(a+1)}+\frac{1}{20(a+1)^2}}\right). \qquad (11)
\end{aligned}
$$

It remains to bound $D_k(P)-k-k'+1$. Since we assumed that P satisfies $\mathcal{A}_{k'}$ we have $D_k(P)=\mathbb{E}[D_k(P)]+O(\sqrt{n}\log n)$, Thus, for $R\leq k\leq n-R$, where $R=n^{1-1/20(a+1)}$, we may use Lemma 5 to obtain

$$
\begin{aligned}
D_k(P)-k-k'+1 &= \Omega(n^{\frac{1}{a+1}}k^{\frac{a}{a+1}}-k)+O\left(\sqrt{n}\log n\right)-\varepsilon' k \\
&= \Omega(n^{1-\frac{1}{10(a+1)}}). \qquad (12)
\end{aligned}
$$

Here we have used the fact that

$$
\begin{aligned}
k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}}-1\right) &\geq n^{1-\frac{1}{20(a+1)}}\left(\left(\frac{n}{n-n^{1-\frac{1}{20(a+1)}}}\right)^{\frac{1}{a+1}}-1\right)\\
&= n^{1-\frac{1}{20(a+1)}}\left((1+O(n^{-\frac{1}{20(a+1)}}))\frac{n^{-\frac{1}{20(a+1)}}}{a+1}\right)\\
&= \Omega(n^{1-\frac{1}{10(a+1)}}).
\end{aligned}
$$

Putting these bounds together we have

$$
\frac{|\mathcal{M}_1|}{|\mathcal{M}_2|} = \exp\left(O(n^{\frac{1}{2(a+1)}+\frac{1}{20(a+1)^2}})\right) \leq \exp\left(O(n^{21/80})\right). \tag{13}
$$

Combining Corollary 12 and the bound for the ratio $|\mathcal{M}_1|/|\mathcal{M}_2|$, we obtain

$$
\begin{aligned}
\mathbb{P}_{\mathcal{M}'(P)}(\overline{\mathcal{A}_{k^*}}) &\leq \mathbb{P}_{\Gamma_M}(\mathcal{A}_{k^*,M})\frac{|\mathcal{M}_1|}{|\mathcal{M}_2|}\\
&\leq 6(a+1)ne^{-n^{1/3}+O(n^{21/80})} = o(n^{-1})
\end{aligned}
$$

which completes the proof. $\square$

Now suppose that we are given an (H,T)-pattern $P \in \mathcal{P}_n^a$ which satisfies $\mathcal{A}_{k^*}$ and $\mathcal{A}_{k'}$ (recall that $\mathcal{A}_{k^*}$ is defined on $\mathcal{M}'(P)$ in the previous lemma). Let $\mathcal{M}^*(P)$ denote the set of all $(a,1)$-matchings in $\mathcal{M}_n^a$ whose (H,T)-patterns agree with P outside $(k,k^*]$. Define the probability $\mathbb{P}_{\mathcal{M}^*(P)}$ on the set $\mathcal{M}^*(P)$ by choosing the matchings uniformly at random. For a set of matchings $\mathcal{M}$, we also denote by $\mathcal{M}$ the event that a matching lies in $\mathcal{M}$. Viewing the events in $\mathcal{M}^*(P)$ in terms of partial (H,T)-patterns weighted by the number of associated $(a,1)$-matchings, the following lemma establishes that this weighting is so uniform that the error in counting only patterns is negligible.

Lemma 14. *For $i=1,2$, let P_i^* be an arbitrary (H,T)-pattern whose corresponding matchings are in $\mathcal{M}^*(P)$, and let $\mathcal{M}_i^*$ be the set of all matchings in $\mathcal{M}^*(P)$ with (H,T)-pattern P_i^*. Then*

$$
\mathbb{P}_{\mathcal{M}^*(P)}(\mathcal{M}_1^*) = (1+o(1))\mathbb{P}_{\mathcal{M}^*(P)}(\mathcal{M}_2^*).
$$

Proof. From the definitions of $\mathcal{M}_1^*$ and $\mathcal{M}_2^*$ it follows immediately that for $i=1,2$

$$
\mathbb{P}_{\mathcal{M}^*(P)}(\mathcal{M}_i^*) = \frac{|\mathcal{M}_i^*|}{|\mathcal{M}^*(P)|}.
$$

This in turn implies that

$$
\frac{\mathbb{P}_{\mathcal{M}^*(P)}(\mathcal{M}_1^*)}{\mathbb{P}_{\mathcal{M}^*(P)}(\mathcal{M}_2^*)} = \frac{|\mathcal{M}_1^*|}{|\mathcal{M}_2^*|}.
$$

We shall proceed in a similar manner as in the proof of Lemma 13. Since P_1^* and P_2^* differ only on $(k,k^*]$ and agree with P outside the interval, we may write

$$\begin{aligned}\frac{|\mathcal{M}_1^*|}{|\mathcal{M}_2^*|} &\le \prod_{i=k+1}^{k^*} \frac{D_{k^*}(P_1^*)-k^*-k+1}{D_k(P_2^*)-k-k^*+1}\\ &\le \exp\left((k^*-k)\frac{h(k,k^*;P)+k^*-k}{D_k(P)-k-k^*+1}\right)\\ &\le \exp\left(O\left(\frac{(\varepsilon^* k)^2(\frac{p}{1-p}+1)}{n^{1-\frac{1}{10(a+1)}}}\right)\right),\end{aligned}$$

where the last line follows from the definition of $\mathcal{A}_{k^*}$ and (12). Now using the upper bound for $\frac{p}{1-p}$ from Proposition 10 we obtain

$$\frac{|\mathcal{M}_1^*|}{|\mathcal{M}_2^*|} = \exp\left(O(n^{-\frac{1}{4(a+1)}})\right) = 1+o(1).$$

□

7 The number of vertices with a particular in-degree

Using Lemma 14, for any (H,T)-pattern P satisfying $\mathcal{A}_{k^*}$ and $\mathcal{A}_{k'}$ the probability $\mathbb{P}_{\mathcal{M}^*(P)}(d_{k+1}=d+1)$ may be calculated as the number of (H,T)-patterns in $(k,k^*]$ with d Hs in $(k,k+1]$ and $h(k,k^*)-d$ Hs in $(k+1,k^*]$ divided by the total number of (H,T)-patterns in $(k,k^*]$ multiplied by an error factor of $(1+o(1))$. The number of ways of distributing d heads in the $a=t(k,k+1)$ spaces made by $a-1$ tails is the same as the number of ways of choosing the positions of $a-1$ tails out of $d+a-1$ possible locations, i.e. $\binom{d+a-1}{a-1}$. Assume that $0 \le d \le n^{\frac{1}{100(a+1)}}$, then using this and similar calculations for the other factors, we may conclude that

$$\mathbb{P}_{\mathcal{M}^*(P)}(d_{k+1}=d+1) = (1+o(1))\frac{\binom{d+a-1}{a-1}\binom{t(k+1,k^*)+h(k,k^*)-d-1}{t(k+1,k^*)-1}}{\binom{t(k,k^*)+h(k,k^*)-1}{t(k,k^*)-1}}, \tag{14}$$

for all k in the range $R \le k \le n-R$, where $R = n^{1-\frac{1}{20(a+1)}}$. Expanding as factorials and using $t(k,k^*)-a = t(k+1,k^*)$ we see that the right hand side of (14) equals $(1+o(1))$ times

$$\binom{d+a-1}{a-1}\frac{(t(k,k^*)-1)_a}{(t(k,k^*)+h(k,k^*)-1)_a}\frac{(h(k,k^*))_d}{(t(k,k^*)+h(k,k^*)-a-1)_d}. \tag{15}$$

Let $t^* := t(k,k^*) = a\varepsilon^* k$ and $h^* := h(k,k^*)$. Since by assumption $\mathcal{A}_{k^*}$ holds, we obtain

$$(h^*)_d = \big(1+O(d/h^*)\big)^d (h^*)^d = \big(1+O\left(\frac{d^2}{\frac{p}{1-p}t^*}\right)\big)(h^*)^d = (1+o(1))(h^*)^d,$$

where in the last line we have made use of Proposition 10. Since a is fixed and $t^* + h^* \geq h^*$, the other factors in (15) may be estimated in a similar fashion, and thus we may write

$$\mathbb{P}_{\mathcal{M}^*(P)}(d_{k+1} = d+1) = (1+o(1))\binom{d+a-1}{a-1}\left(\frac{t^*}{t^*+h^*}\right)^a \left(\frac{h^*}{t^*+h^*}\right)^d.$$

As one would expect, this is just what one would obtain in the binomial model where heads and tails are independent and the probability of getting a tail T is $\frac{t^*}{t^*+h^*}$. Now, since $\mathcal{A}_{k^*}$ holds, we can substitute the expression in (9) for h^* and so we have

$$\begin{aligned}
\left(\frac{h^*}{t^*+h^*}\right)^d &= \left(\frac{(1+O(\delta))a\varepsilon^* k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}} - 1\right)}{a\varepsilon^* k + (1+O(\delta))a\varepsilon^* k\left(\left(\frac{n}{k}\right)^{\frac{1}{a+1}} - 1\right)}\right)^d \\
&= \left((1+O(\delta))\frac{\left(\frac{n}{k}\right)^{\frac{1}{a+1}} - 1}{\left(\frac{n}{k}\right)^{\frac{1}{a+1}}}\right)^d \\
&= (1+O(\delta d))\left(1 - \left(\frac{k}{n}\right)^{\frac{1}{a+1}}\right)^d.
\end{aligned}$$

Similarly

$$\left(\frac{t^*}{t^*+h^*}\right)^a = (1+O(\delta))\left(\frac{k}{n}\right)^{\frac{a}{a+1}}.$$

Since $\delta = n^{-1/90(a+1)}$ and $0 \leq d \leq n^{1/100(a+1)}$, both $O(\delta)$ and $O(\delta d)$ are $o(1)$. Thus putting all these calculations together and letting $\kappa = k/n$ for $R \leq k \leq n - R$, we get

$$\mathbb{P}_{\mathcal{M}^*(P)}(d_{k+1} = d+1) = (1+o(1))\binom{d+a-1}{a-1}\kappa^{\frac{a}{a+1}}\left(1 - \kappa^{\frac{1}{a+1}}\right)^d. \tag{16}$$

Now, we need to go back to the probability space $(\mathcal{M}_n^a, \mathbb{P})$ of all $(a,1)$-matchings. We do this in two steps. First we shall transfer equation (16) to the probability space $\mathcal{M}'(P)$. Given any (H,T)-pattern P' satisfying $\mathcal{A}_{k'}$, let $\mathcal{P}^*$ be the set of all $(a,1)$-matchings in $\mathcal{M}'(P')$ which satisfy the event $\mathcal{A}_{k^*}$ (and thus also $\mathcal{A}_{k'}$). We further partition $\mathcal{P}^*$ into maximal disjoint subsets $\mathcal{P}_i^*$ so that for each i there is a pattern P_i so that all the matchings in $\mathcal{P}_i^*$ have patterns whose restrictions to the complement of $(k, k^*]$ are the same as that of P_i. We also partition the space $\mathcal{M}'(P')$ into $\mathcal{P}^*$ and its complement $\overline{\mathcal{P}^*}$. For a set of matchings $\mathcal{P}$, we again denote by $\mathcal{P}$ the event that a matching lies in $\mathcal{P}$. Now using the law of total probability together with Lemma 13, and applying (16) to each of the P_i^*, we obtain

$$\begin{aligned}
\mathbb{P}_{\mathcal{M}'(P')}(d_{k+1} = d+1) &= \mathbb{P}_{\mathcal{M}'(P')}(d_{k+1} = d+1 \mid \overline{\mathcal{P}^*})\mathbb{P}_{\mathcal{M}'(P')}(\overline{\mathcal{P}^*}) \\
&\quad + \sum_{\mathcal{P}_i^* \subseteq \mathcal{P}^*} \mathbb{P}_{\mathcal{M}'(P')}(d_{k+1} = d+1 \mid \mathcal{P}_i^*)\mathbb{P}_{\mathcal{M}'(P')}(\mathcal{P}_i^*) \\
&= o(n^{-1}) + \sum_{\mathcal{P}_i^* \subseteq \mathcal{P}^*} \mathbb{P}_{\mathcal{M}^*(P_i)}(d_{k+1} = d+1)\mathbb{P}_{\mathcal{M}'(P')}(\mathcal{P}_i^*) \\
&= o(n^{-1}) + (1+o(1))\binom{d+a-1}{a-1}\kappa^{\frac{a}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d.
\end{aligned}$$

For the second step, partition the primary probability space $\mathcal{M}_n^a$ into the set $\mathcal{P}'$, of those matchings whose patterns satisfy the event $\mathcal{A}_k'$ and its complement $\overline{\mathcal{P}'}$. Then by a similar argument as above, this time partitioning $\mathcal{P}'$ according to the restrictions of the patterns to the complement of $(k, k']$ and applying Corollary 9, we have

$$\mathbb{P}(d_{k+1} = d+1) = o(n^{-1}) + (1+o(1))\binom{d+a-1}{a-1}\kappa^{\frac{a}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d.$$

Letting $a = 1$, one sees that this is a generalization of the result found in [9] for that special case.

We may now calculate the expected number $\mathbb{E}[N_n^1(d)]$ of vertices with in-degree d in G_n^1 as follows.

$$\begin{aligned}
\mathbb{E}[N_n^1(d)] &= \sum_{k=0}^{n-1} \mathbb{P}(d_{k+1} = d+1) \\
&= o(1) + O(R) + (1+o(1))\binom{d+a-1}{a-1}\sum_{k=R}^{n-R}\kappa^{\frac{a}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d.
\end{aligned}$$

We will show that $\frac{1}{n}\mathbb{E}[N_n^1(d)] = \Theta\left(d^{-2-a}\right)$ for large d. In fact, we will determine this quantity more accurately. To do this, we shall approximate the last sum as an integral. Let $f(\kappa) = \kappa^{\frac{a}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d$. Proceeding in a similar way as in [9], we see that f is unimodal and so as n tends to infinity, the proportional change in f as κ varies by $1/n$ is then given by $(1+o(1))$ times

$$\frac{1}{nf(\kappa)}\frac{\partial f}{\partial \kappa} = \frac{a}{(a+1)n\kappa} - \frac{d}{(a+1)n\kappa^{\frac{a}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)}.$$

Since $n\kappa^{a/(a+1)}\left(1-\kappa^{1/(a+1)}\right)/d$ tends to infinity for $n^{-1/20(a+1)} \leq \kappa \leq 1 - n^{-1/20(a+1)}$ and $0 \leq d \leq n^{1/100(a+1)}$, this proportional change is $o(1)$ uniformly in κ. Therefore the sum above may be written as

$$(1+o(1))n\int_{(R+1)/n}^{n-R/n}\kappa^{\frac{a}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d d\kappa = (1+o(1))n\int_0^1\kappa^{\frac{a}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d d\kappa.$$

The exact solution of the integral is

$$(a+1)(2a)!\frac{d!}{(d+2a+1)!} = \Theta\left(d^{-2a-1}\right).$$

To see this, we first perform the substitution $\kappa = (1-u)^{a+1}$. After obtaining the new form of the integral, it remains to prove the following identity.

$$\int_0^1 (1-u)^{2a} u^d du = (2a)! \frac{d!}{(d+2a+1)!}. \tag{17}$$

This we do by induction on a. The base case for the induction is $a = 1$, which was studied by Bollobás et al. [9]; simply by multiplying out and integrating one obtains

$$\int_0^1 (1-u)^2 u^d du = \frac{2}{(d+1)(d+2)(d+3)}.$$

The induction step is easy to establish by applying the induction hypothesis to each term on the right of

$$\int_0^1 (1-u)^{2(a+1)} u^d du = \int_0^1 (1-u)^{2a} u^d du - 2\int_0^1 (1-u)^{2a} u^{d+1} du + \int_0^1 (1-u)^{2a} u^{d+2} du.$$

By cancelling the factors in the factorials which contain d and taking the factor d^{a-1} in front of the sum into account, we see that for $d \le n^{1/100(a+1)}$ and $r = 1$, $\frac{1}{n}\mathbb{E}[N_n^1(d)] = (1+o(1))P_{in}(d)$, where $P_{in}(d)$ is defined as in Theorem 1.

We now return to the general case $r \ge 1$. Recall that G_n^r is formed from G_{rn}^1 by merging consecutive sets of r vertices. Our aim is to calculate the probability $\mathbb{P}(d_{k+1}^r = d + r)$, where d_{k+1}^r denotes the degree of the $(k+1)^{th}$ vertex of G_n^r. To do this, we calculate the probability of the equivalent event in G_{rn}^1. If we let $K = rk$, $K^* = rk^*$ and $N = rn$ then proceeding as we did for the case $r = 1$, this probability may be calculated as the number of (H,T)-patterns in $(K, K^*]$ with exactly d Hs before the occurrence of the $(ar)^{th}$ T (thus $(K, K+r]$ consists of exactly $d + ar$ points, with d heads distributed among ar spaces made by the tails) and $h(K, K^*) - d$ Hs in $(K+r, K^*]$, divided by the total number of (H,T)-patterns in $(K, K^*]$. Thus by the same argument as in the case $r = 1$ we have the following, which immediately implies Theorem 2. Indeed, for any $a, r \in \mathbb{N}$ and any k with $n^{1-1/20(a+1)} \le k \le n - n^{1-1/20(a+1)}$ and $0 < d \le n^{1/100(a+1)}$, we have

$$\mathbb{P}(d_{k+1}^r = d + r) = o(n^{-1}) + (1+o(1))\binom{d+ar-1}{ar-1}\kappa^{\frac{ra}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d,$$

where $\kappa = k/n$. Since $K/N = k/n$, the calculation of $\mathbb{E}[N_n^r(d)]$ barely differs from that of $\mathbb{E}[N_n^1(d)]$ above. This time the appropriate integral to approximate the sum is

$$\int_0^1 \kappa^{\frac{ra}{a+1}}\left(1-\kappa^{\frac{1}{a+1}}\right)^d d\kappa,$$

which has the exact solution

$$(a+1)(ar+a)!\frac{d!}{(d+ar+a+1)!} = \Theta\left(d^{-ar-a-1}\right).$$

As before, we shall prove this by induction; this time on $a \in \mathbb{N}$ and $r \in \mathbb{N}$. Again, performing the substitution $\kappa = (1-u)^{a+1}$ one sees that it suffices to establish the identity

$$\int_0^1 (1-u)^{ar+a} u^d du = (ar+a)!\frac{d!}{(d+ar+a+1)!}. \tag{18}$$

When $r = 1$, this reduces to the previous case (17), which we have already proven for all $a \geq 1$. Thus assume that for some $a, r \in \mathbb{N}$, (18) holds. Then for $r+1$, using the binomial expansion, we obtain

$$\begin{aligned} \int_0^1 (1-u)^{a(r+1)+a} u^d du &= \int_0^1 (1-u)^a (1-u)^{ar+a} u^d du \\ &= \sum_{i=0}^{a} (-1)^i \binom{a}{i} \int_0^1 (1-u)^{ar+a} u^{d+i} du \qquad (19) \\ &= (ar+a)! \sum_{i=0}^{a} (-1)^i \binom{a}{i} \frac{(d+i)!}{(d+ar+a+i+1)!}, \qquad (20) \end{aligned}$$

where, just as in the case $r = 1$, we have applied the induction hypothesis (18) to each integral on the right of (19). We now make use of the following identity (see for example [5] or [22]),

$$\sum_{i=0}^{a} (-1)^i \binom{a}{i} \frac{(x+i)!}{(y+i)!} = \frac{x!(y-x+a-1)!}{(y+a)!(y-x-1)!}$$

which holds for all $x, y \in \mathbb{N}$, where $x < y$. Therefore, letting $x = d$ and $y = d + a(r+1) + 1$, we may conclude that

$$\int_0^1 (1-u)^{a(r+1)+a} u^d du = (a(r+1)+a)! \frac{d!}{(d+a(r+1)+a+1)!},$$

which completes the induction.

Now by multiplying by the factor d^{ar-1} in front of the integral, we obtain $\Theta\left(d^{-2-a}\right)$ as before. We have thus shown that the expected number of vertices $\mathbb{E}[N_n^r(d)]$ in G_n^r with in-degree d is given by

$$\frac{1}{n}\mathbb{E}[N_n^r(d)] = (1+o(1))(a+1)(ar+a)! \binom{d+ar-1}{ar-1} \frac{d!}{(d+ar+a+1)!}.$$

for each fixed $r \in \mathbb{N}$ and $a \in \mathbb{N}$. We are now finally in a position to prove Theorem 1.

Proof of Theorem 1. We turn our interest now to the graph G_n^r, $n \geq 1$ as one graph from the graph process $(G_t^r)_{t \geq 0}$, where $r \in \mathbb{N}$ is fixed. With a view to using the Azuma-Hoeffding inequality (Lemma 4), we define the martingale $X_t = \mathbb{E}[N_n^1(d) \mid G_t^r]$, where $0 \leq d \leq n^{1/100(a+1)}$ as before. Then $X_0 = \mathbb{E}[N_n^r(d)]$ and $X_n = N_n^r(d)$. It remains to establish the bounded differences property. We claim that $|X_t - X_{t-1}| \leq 2r$ for all t. Proceeding as in [9], consider what happens at stage t of the graph process. Whether vertex t sends r edges to the set of vertices $\{u_1, ..., u_r\}$ or to $\{v_1, ..., v_r\}$ does not effect the joint degree distribution of any of the other vertices. Thus the difference in the expected number of vertices with in-degree d is at most $2r$. Applying the Azuma-Hoeffding inequality, we obtain for all $0 \leq d \leq n^{1/100(a+1)}$ that

$$\mathbb{P}\left(\mid N_n^r(d) - \mathbb{E}[N_n^r(d)] \mid \geq r\sqrt{n \log n}\right) \leq e^{-\frac{\log n}{8}} = n^{-1/8}.$$

Since $\mathbb{E}[N_n^r(d)] = \Theta\left(nd^{-2-a}\right)$ is far larger than $r\sqrt{n \log n}$ in the range of d given above, the required result follows. □

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