CHAPTER 3

More General Methods

This chapter develops a more general view of the class of problems that are typified by the traveling-salesman problem. The resulting theory of subadditive Euclidean functionals turns out to offer a rewarding approach to many concrete problems and also gets closer to the essential features that make possible theorems like that of Beardwood, Halton, and Hammersley. The chapter also reviews recent progress on rates of convergence that have been made possible by the consideration of two-sided bounds.

3.1. Subadditive Euclidean functionals.

We begin by detailing some general properties of a function L from the set of finite subsets of $\mathbb{R}^d$ to the nonnegative real numbers $\mathbb{R}^+$. The intention of these properties is to echo the most basic features of the TSP tour-length function. We first impose a natural normalization,

$$L(\emptyset) = 0, \tag{3.1}$$

and then we consider only the simplest geometric properties of homogeneity and translation invariance:

$$L(\alpha x_1, \alpha x_2, \dots, \alpha x_n) = \alpha L(x_1, x_2, \dots, x_n) \quad \text{for all } \alpha > 0 \tag{3.2}$$

and

$$L(x_1 + y, x_2 + y, \dots, x_n + y) = L(x_1, x_2, \dots, x_n) \quad \text{for all } y \in \mathbb{R}^d. \tag{3.3}$$

of the expected tour lengths, but since the property is crucial in the abstract setting, it deserves a special christening.

Geometric subadditivity hypothesis. There exists a constant C_0 such that for all integers $m \geq 1$, $n \geq 1$, and $\{x_1, x_2, \ldots, x_n\} \subset [0,1]^d$, we have

$$(3.4) \qquad L(\{x_1, x_2, \ldots, x_n\}) \leq \sum_{i=1}^{m^d} L(\{x_1, x_2, \ldots, x_n\} \cap Q_i) + C_0 m^{d-1},$$

where $\{Q_i\}$, $1 \leq i \leq m^d$, is the partition of $[0,1]^d$ into cubes of edge length $1/m$.

On many occasions, geometric subadditivity ((3.4)) can be verified at once for all $m \geq 1$, but in some instances we build our way from the case of $m = 2$ to the general case. For the moment, we just recall that in our analysis of the TSP, we found (3.4) directly for all m at one time, and this situation may be typical. Still, we will see later that there is a benefit to having a few tools around to help prove geometric subadditivity.

In addition to properties (3.1)–(3.4), there is one further property of the TSP functional that proved useful in our earlier analysis. The TSP functional is *monotone* in the sense that for all n and $\{x_i\}$, we have

$$(3.5) \qquad L(x_1, x_2, \ldots, x_n) \leq L(x_1, x_2, \ldots, x_n, x_{n+1}).$$

This last property is evident for the TSP, but as we will see shortly, the property is not present in a number of closely related problems that are of considerable importance in the theory of combinatorial optimization. We will revisit this issue of monotonicity in a subsequent section, but for the moment we will exploit monotonicity as best we can.

Euclidean functionals that satisfy (3.4) will be called *subadditive Euclidean functionals,* and the analysis of such functionals is at the heart of this chapter. If (3.5) also holds, we say that L is a *monotone subadditive Euclidean functional,* and this is one particularly simple class of processes that seems to go a long way in capturing the features of the TSP that provide for an effective asymptotic analysis. The main aim of this first section is to show that properties (3.1)–(3.5) are sufficient to determine the asymptotic behavior of $L(X_1, X_2, \ldots, X_n)$, where the $\{X_i\}$'s are independent and uniformly distributed on $[0,1]^d$.

THEOREM 3.1.1 (basic theorem of subadditive Euclidean functionals). *Suppose L is a monotone subadditive Euclidean functional. If the random variables $\{X_i\}$ are independent with the uniform distribution on $[0,1]^d$, then as $n \to \infty$ we have with probability one that*

$$L(X_1, X_2, \ldots, X_n)/n^{(d-1)/d} \to \beta_L,$$

where $\beta_L \geq 0$ is a constant.

Proof. We first check that our assumptions guarantee that the worst-case bound $\|L(X_1, X_2, \ldots, X_n)\|_\infty$ does not grow too rapidly. With even a modest bound on this L^∞ norm, we will be at liberty to use Poissonization and to consider means and variances at will.

The argument for our bound is based on induction on the cardinality of the finite set $F = \{x_1, x_2, \ldots, x_n\} \subset [0,1]^d$. To set up the induction, we first take

$A = C_0 2^{d-1}$, where C_0 is the constant of (3.4), and take $B = A + L(\{x\})$, where we note that by translation invariance ((3.3)), the value of $L(\{x\})$ does not depend on the value of $x \in [0,1]^d$. By these choices, we certainly have that for $F \subset [0,1]^d$ with $\operatorname{card} F = 1$,

$$(3.6) \qquad L(F) \leq B \operatorname{card}(F) - A.$$

We take (3.6) as our induction hypothesis, and we assume that (3.6) holds for all $F \subset [0,1]^d$ with $1 \leq \operatorname{card} F < n$.

We now consider the partition of $[0,1]^d$ into 2^d equal subcubes Q_i with edge length $\frac{1}{2}$, and we let $F \subset [0,1]^d$ be any set with $\operatorname{card} F = n$. By the translation property (3.3), we can assume without loss of generality that F is not contained in any one of the Q_i's, so if we let $I = \{i : F \cap Q_i \neq \emptyset\}$, we can assume that $\operatorname{card} I \geq 2$. By geometric subadditivity (3.4) and the induction hypothesis, we therefore find that

$$(3.7) \quad L(F) \leq \sum_{i \in I} L(F \cap Q_i) + C_0 2^{d-1} \leq Bn - 2A + C_0 2^{d-1} = Bn - A,$$

so (3.7) completes the proof of the induction step.

Now we are in a position to take advantage of Poissonization in a way that parallels our analysis of the TSP. We let Π denote the Poisson process in $\mathbb{R}^d$ with unit intensity, and we set $Z(t) = L(\Pi[0,t]^d)$. We note that by our induction argument we have $Z(t) = O(\operatorname{card}(\Pi[0,t]^d))$, so $Z(t)$ has moments of all orders.

We first work toward showing that $EZ(t)/t^d$ converges, and for a while, the argument closely parallels one from the previous chapter. By (3.4) applied to $\Pi[0,t]^d$, we again get $EZ(t) \leq m^d EZ(t/m) + C_0 t m^{d-1}$, which on replacing t by mt and dividing by $m^d t^d$ gives us the key relation:

$$(3.8) \qquad EZ(mt)/(m^d t^d) \leq EZ(t)/t^d + C_0 t^{1-d}.$$

We can then define γ by

$$(3.9) \qquad 0 \leq \gamma \equiv \liminf_{m \to \infty} EZ(m)/m^d \leq EZ(1) + C_0 < \infty.$$

Now for the last part of the argument that parallels our analysis of the TSP, we note that given any $\epsilon > 0$, we can choose a t_0 such that

$$EZ(t_0)/t_0^d + C_0 t_0^{1-d} \leq \gamma + \epsilon.$$

The argument now changes at least a little as we start to rely more on the monotonicity property (3.5) of L, though the significant changes emerge only when we look at the second moment of $Z(t)$ and when we need to back out of the Poissonization.

We first note that the monotonicity (3.5) of L gives us the pathwise monotonicity of $Z(t)$, so for $mt_0 \leq u < (m+1)t_0$ we certainly have the expectation bounds

$$EZ(u)/u^d \leq EZ((m+1)t_0)/(mt_0)^d \leq (\gamma + \epsilon)(m+1)^d/m^d.$$

Hence we find by taking the limit supremum and using the arbitrariness of m that

$$\limsup_{u\to\infty} EZ(u)/u^d \le \liminf_{t\to\infty} EZ(t)/t^d + \epsilon,$$

so by the arbitrariness of $\epsilon > 0$, we conclude that

$$EZ(t)/t^d \to \gamma \quad \text{as } t \to \infty. \tag{3.10}$$

The next step is to work toward an understanding of the second moment of $Z(t)$, and finally the argument sets a new course. We begin by applying the geometric subadditivity condition (3.4) with $m = 2$ to the Poisson sample $\Pi[0,2t]^d$. On writing $Z_i(t)$ for $L(\Pi(Q_i))$ and noting the change of scale, we find

$$Z(2t) \le \sum_{i=1}^{2^d} Z_i(t) + C_0 2^{d-1} t. \tag{3.11}$$

To get a somewhat simpler inequality, we set $\hat{Z}(t) = Z(t) + 2C_0 t$ and $\hat{Z}_i(t) = Z_i(t) + 2C_0 t$ so that inequality (3.11) implies

$$\hat{Z}(2t) \le \sum_{i=1}^{2^d} \hat{Z}_i(t). \tag{3.12}$$

If we write $\phi(t) = E\hat{Z}(t) = E\hat{Z}_i(t)$ and $\psi(t) = (E\hat{Z}(t)^2)^{\frac{1}{2}} = (E\hat{Z}_i(t)^2)^{\frac{1}{2}}$, we find by squaring (3.12) and taking expectations that

$$\psi^2(2t) \le 2^d \psi^2(t) + 2^d(2^d - 1)\phi^2(t).$$

Introducing $V(t) = \operatorname{Var} Z(t) = \psi^2(t) - \phi^2(t)$, we are led to

$$V(2t) = \psi^2(2t) - \phi^2(2t) \le 2^d V(t) + 2^{2d}\phi^2(t) - \phi^2(2t),$$

and upon dividing by $(2t)^{2d}$, we find our fundamental recursion:

$$\frac{V(2t)}{(2t)^{2d}} - \frac{V(t)}{2^d t^{2d}} \le \frac{\phi^2(t)}{t^{2d}} - \frac{\phi^2(2t)}{(2t)^{2d}}. \tag{3.13}$$

By applying this bound for $t, 2t, \ldots, 2^{M-1}t$ and summing, we find

$$\sum_{k=1}^{M} \frac{V(2^k t)}{(2^k t)^{2d}} - 2^{-d} \sum_{k=0}^{M-1} \frac{V(2^k t)}{(2^k t)^{2d}} \le \frac{\phi^2(t)}{t^{2d}},$$

so finally,

$$(1 - 2^{-d}) \sum_{k=1}^{M} \frac{V(2^k t)}{(2^k t)^{2d}} \le \frac{\phi^2(t)}{t^{2d}} + \frac{V(t)}{t^{2d}}.$$

Since this bound holds for all M, we can let $M \to \infty$ and arrive at the main fact we need concerning $V(t)$:

$$\sum_{k=1}^{\infty} V(2^k t)/(2^k t)^{2d} < \infty. \tag{3.14}$$

In order to use (3.14) to complete the proof, we again call on the Borel–Cantelli lemma, but this time we need to add an interpolation argument that makes essential use of monotonicity; after all, the sum (3.14) is over a very sparse subsequence.

The argument first requires that we bring our original variables $\{X_1, X_2, \ldots\}$ back into view. To do this, we let $N(t)$ be a regular (one-dimensional) Poisson counting process with rate one that is independent of $\{X_i : 1 \le i < \infty\}$. The point of this change is the simple observation that $Z(t) = L(\Pi[0,t]^d)$ and $tL(X_1, X_2, \ldots, X_{N(t^d)})$ have the same distribution for each t, but we have made progress since the $\{X_i\}$'s of our theorem are present in the second expression. Since we have already found that $EZ(t) \sim \beta t^d$ as $t \to \infty$, we find from (3.14) and Chebyshev's inequality that for any $\epsilon > 0$ we have

$$\sum_{k=0}^{\infty} P\{|t2^k L(X_1, X_2, \ldots, X_{N(t^d 2^{kd})})/(t2^k)^d - \beta| > \epsilon\} < \infty.$$

By the Borel–Cantelli lemma, we see that for each $t > 0$,

$$\lim_{k\to\infty} L(X_1, X_2, \ldots, X_{N(t^d 2^{kd})})/(t2^k)^{d-1} = \beta \quad \text{a.s.} \tag{3.15}$$

Now for the interpolation argument. For any fixed integer $p > 0$ and any real number $s \ge 2^p$, we can find integers t and k such that $2^p \le t < 2^{p+1}$ and $2^k t \le s \le 2^k(t+1)$. By the monotonicity of L, we then have

$$L(X_1, X_2, \ldots, X_{N(t^d 2^{kd})}) \le L(X_1, X_2, \ldots, X_{N(s^d)}) \le L(X_1, X_2, \ldots, X_{N((t+1)^d 2^{kd})}).$$

Since p is fixed, the set of integers $\{t : 2^p \le t < 2^{p+1}\}$ is finite; the last pair of inequalities and the subsequence limit (3.15) imply that with probability one we have for real $s \to \infty$ that

$$\limsup_{s\to\infty} L(X_1, X_2, \ldots, X_{N(s^d)})/s^{d-1} \le \beta(1 + 2^{-p})^{d-1}$$

and

$$\liminf_{s\to\infty} L(X_1, X_2, \ldots, X_{N(s^d)})/s^{d-1} \ge \beta(1 + 2^{-p})^{1-d}.$$

By the arbitrariness of the integer p, we see as $s \to \infty$ that

$$\lim_{s\to\infty} L(X_1, X_2, \ldots, X_{N(s^d)})/s^{d-1} = \beta \quad \text{a.s.} \tag{3.16}$$

The last step we take is to back out of the Poisson indexing. To do so, we let $\tau(n) = \min\{t : N(t^d) = n\}$ and note that by the definition of $\tau(n)$ we have the identity

$$\begin{aligned} &L(X_1, X_2, \ldots, X_n)/n^{(d-1)/d} \\ &= \{L(X_1, X_2, \ldots, X_{N(\tau(n)^d)})/\tau(n)^{d-1}\}\{\tau(n)^{d-1}/n^{(d-1)/d}\}. \end{aligned} \tag{3.17}$$

Finally, the first factor above goes to β with probability one by (3.16), and a well-known property of the Poisson process is that $\tau(n)/n^{1/d} \to 1$ with probability one; so the second factor also converges to one almost surely, and we see that the proof of the theorem is complete.