Percolation, connectivity, coverage and colouring of random geometric graphs

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Abstract In this review paper, we shall discuss some recent results concerning several models of random geometric graphs, including the Gilbert disc model G_r, the k-nearest neighbour model $G_k^{\rm nn}$ and the Voronoi model $G_{\mathcal{P}}$. Many of the results concern finite versions of these models. In passing, we shall mention some of the applications to engineering and biology.

1 Introduction

Place a million points uniformly at random in a large square and connect every point to the six points closest to it. What can we say about the resulting graph? Is it connected, and, if not, does it contain a connected component with at least a hundred thousand vertices? In this paper, we consider such questions for some of the most natural models of a random geometric graph, including the one above. From a practical point of view, these graphs are excellent models for *ad-hoc wireless networks*, in which some radio transceivers lie scattered over a large region, and where each transceiver can only communicate with a few others nearby. From a more mathematical standpoint, the models act as a bridge between the theory of classical random graphs [17]

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and that of percolation [18], and our study of them will draw inspiration and use tools from both these more established fields. (It is tempting to write "older" fields, but in fact Gilbert's pioneering papers [31, 32] appeared only shortly after those of Broadbent and Hammersley on percolation, and those of Erdős and Rényi on random graphs.)

For all the models, we will take the vertex set to be a unit intensity *Poisson process* $\mathcal{P}$ in $\mathbb{R}^2$, or the restriction of $\mathcal{P}$ to a square, so it is convenient to make some remarks about this at the outset. All our finite results will carry over to the case of uniformly distributed points, but for their proofs, and for the *statements* of our infinite results, it is far better to use a Poisson process.

There are several ways of defining such a process, of which the following is perhaps the easiest to describe. First recall that a *Poisson random variable* of mean λ is a discrete random variable X for which

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

As is usual, we will denote this by $X \sim \mathrm{Po}(\lambda)$. Now tessellate $\mathbb{R}^2$ with unit squares, and consider a family (X_i) of independent random variables, indexed by the squares, where each $X_i \sim \mathrm{Po}(1)$. We place X_i points uniformly at random in the square i. The result is a unit intensity Poisson process $\mathcal{P}$.

One of the key features of this model is its *independence*: the number of points of $\mathcal{P}$ in a measurable region $A \subset \mathbb{R}^2$ is a Poisson random variable with mean $|A|$ (the Lebesgue measure of A), regardless of what is happening outside A. Moreover, conditioned on there being k points in A, their distribution is uniform. See [18] for more background.

Throughout the paper, the phrase "with high probability" will mean "with probability tending to one as $n \to \infty$". Also, all logarithms in this paper are to the base e.

2 The Gilbert disc model

Our first model was introduced and studied by E.N. Gilbert in 1961 [31], and has since become known as the *disc model* or *Gilbert model*. To define it, fix $r > 0$, let $\mathcal{P}$ be a Poisson process of intensity one in the plane $\mathbb{R}^2$, and connect two points of $\mathcal{P}$ by an edge if the distance between them is less than r. Denote the resulting infinite graph by G_r.

2.1 Percolation

Although Gilbert's main focus was the study of communications networks, he noted that G_r could also model the spread of a contagious disease. For the

second application, and perhaps also for the first, one is primarily interested in *percolation*, in the following sense. Let us suppose that, without loss of generality, the origin is one of the points of $\mathcal{P}$. Writing $a = \pi r^2$, and $\theta(a)$ for the probability that the origin belongs to an infinite connected component of G_r, Gilbert defined the *critical area* a_c as

$$a_c = \sup\{a : \theta(a) = 0\}.$$

In other words, for $a > a_c$, there is a non-zero probability that the disease spreads, or that communication is possible to some arbitrarily distant nodes of the network. In this case we say that the model *percolates*. Since $\theta(a)$ is clearly monotone, $\theta(a) = 0$ if $a < a_c$. Currently the best known bounds, due to Hall [36], are

$$2.184 \leq a_c \leq 10.588,$$

although in a recent paper Balister, Bollobás and Walters [13] used 1-independent percolation to show that, with confidence 99.99%,

$$4.508 \leq a_c \leq 4.515,$$

which is consistent with the non-rigorous bounds

$$4.51218 \leq a_c \leq 4.51228$$

obtained by Quintanilla, Torquato and Ziff [50].

We can consider the same problem in d dimensions, where we use balls of volume v rather than discs, and define $\theta^d(v)$ and v_c^d in the obvious manner. Penrose [47] proved the following.

Theorem 1.

$$v_c^d \to 1 \text{ as } d \to \infty.$$

Most of the volume in a high-dimensional ball is close to the boundary, and hence one might expect that the same conclusion holds for a two-dimensional annulus where the ratio of the inner and outer radii tends to 1. This is indeed true, and was proved independently by Balister, Bollobás and Walters [12] and Franceschetti, Booth, Cook, Meester and Bruck [28]. However, as shown in [12], the corresponding result for square annuli is false. A general condition under which the critical area tends to 1 is given in [14].

2.2 Connectivity

Penrose [48, 49] considered the following finite version of G_r. For this, we only consider points of $\mathcal{P}$ lying in a fixed square S_n of area n, again joining two points if the distance between them is less than r. Penrose proved the following result on the connectivity of the resulting model $G_r(n)$.

Theorem 2. *If* $\pi r^2 = \log n + \alpha$ *then*

$$\mathbb{P}(G_r(n) \text{ is connected}) \to e^{-e^{-\alpha}}.$$

In particular, the above probability tends to 1 *iff* $\alpha \to \infty$.

This result has an exact analogue in the theory of classical random graphs [17]. Indeed, in both cases the obstructions to connectivity are isolated vertices. In fact, for both models, it is not hard to calculate the expected number of isolated vertices, and then to show that their number has a distribution that is approximately Poisson. The hard part is to show that there are no other obstructions. For $G_r(n)$, Penrose first shows that the obstructions must be small, that is, of area at most $C \log n$ (with our normalization). This he achieves by *discretization*. Since many of the proofs of the theorems we will discuss use this technique, we give a brief account of it, for this case. The basic idea is to tessellate our large square with smaller squares of side length $r/\sqrt{5}$. Any component in $G_r(n)$ must be surrounded by a connected path consisting of, say, l *vacant* squares, none of which can contain any points of $\mathcal{P}$. Even though the number of such paths of squares is exponential in l, if the component is large (so that $l \geq K$ for some absolute constant K), it is not hard to show that the probability of such a vacant path existing anywhere in S_n tends to zero. Thus, with high probability, if $G_r(n)$ is disconnected, it contains a small component. Penrose completes his proof with a delicate local argument, showing that, for the relevant range of values of r, this small component is, with high probability, an isolated vertex.

As shown by Penrose, the story for k-connectivity also mirrors that for classical random graphs, in that the principal obstructions are vertices of degree exactly $k-1$. For detailed statements and proofs, the reader is referred to [49].

There are various ways to generalize this model. One, treated thoroughly in [43], is to choose the disc radii to be independent and identically distributed random variables. Another possibility is to keep the radii fixed at r, but vary the intensity of the underlying Poisson process. In one such model, suggested by Etherington, Hoge and Parkes [24], the intensity $\rho(x)$ of $\mathcal{P}$ at distance x from the origin is given by a gaussian distribution, so that

$$\rho(x) = \tfrac{n}{\pi} e^{-x^2}.$$

This model $G_r^{\text{Gauss}}(n)$ was analyzed in detail by Balister, Bollobás, Sarkar and Walters [8], who determined the threshold for connectivity.

Theorem 3. *If*

$$2r\sqrt{\log n} = \log\log n - \tfrac{1}{2}\log\log\log n + f(n),$$

then, with high probability, $G_r^{\text{Gauss}}(n)$ *is connected if* $f(n) \to \infty$ *and disconnected if* $f(n) \to -\infty$.

2.3 Coverage

For most of the remainder of this section, we will imagine that the points of our Poisson process $\mathcal{P}$ are sensors designed to monitor a large square region S_n of area n. Such monitoring is feasible if the sensing discs $D_r(p)$ cover S_n, so that

$$S_n \subset \bigcup_{p \in \mathcal{P}} D_r(p).$$

How large should we make $r = r(n)$ so that this occurs with high probability?

Before turning to recent results, we consider the original application of Moran and Fazekas de St Groth [45]. They considered the problem of covering the surface of a sphere with circular caps, and write:

> This problem arises in practice in the study of the theory of the manner in which antibodies prevent virus particles from attacking cells. Thus an influenza virus may be considered to be a sphere of radius about 40 mμ. Antibodies are supposed to be cigar-shaped molecules of length about 27 mμ and of a thickness which will be neglected. The antibodies are assumed to attach themselves at their ends to the virus particle, standing up rigidly on the surface and thus shielding a circular area on the virus from possible contact with the surface of a cell.

Also noteworthy is their simulation method:

> ...an experiment was carried out using table tennis balls. These had a mean diameter of 37.2mm. with a standard deviation around this mean of 0.02mm. One hundred holes of diameter 29.9mm. were punched in an aluminium sheet forming one side of a flat box. The balls were held firmly against the holes by a foam rubber pad, and sprayed with a duco paint. After drying they were removed and replaced at random by hand. Forty sprayings were done in each of three sets of 100 balls.

This was also one of the problems considered by Gilbert [32], who performed *his* simulations on an IBM 7094 computer. His paper contains the following critical observation, which we will state in the context of our original formulation of the problem. For the (open) discs $D_r(p)$ to cover S_n, it is not only necessary but also sufficient that the following three conditions hold:

- Every intersection of 2 disc boundaries inside S_n is covered by a third disc
- Every intersection of a disc boundary with ∂S_n is covered by a second disc
- There is at least one such intersection (of either type)

Hall [35] used this observation to establish the following criterion.

Theorem 4. *If*

$$\pi r^2 = \log n + \log\log n + f(n),$$

then a necessary and sufficient condition for the discs $D_r(p)$ to cover S_n with high probability is that $f(n) \to \infty$.

The proof proceeds by showing that if r is as in the statement of Theorem 4, then the expected number of uncovered intersections is asymptotically $4e^{-f(n)}$. Thus if $f(n) \to \infty$, by Gilbert's observation, we obtain coverage with high probability. For the other direction, Hall applies the second moment method (to the uncovered *area*).

Slightly later, Janson [38] obtained very general results on the probability of coverage. For our case, his result is as follows.

Theorem 5. *If*

$$\pi r^2 = \log n + \log\log n + x,$$

then as $n \to \infty$

$$\mathbb{P}(S_n \text{ is covered}) \to e^{-e^{-x}}.$$

Recently, a shorter proof of Theorem 5, with bounds on the error term, was obtained by Balister, Bollobás, Sarkar and Walters [11]. The idea is that, while the uncovered intersections occur in groups, these groups consist simply of the intersections bordering the uncovered *regions*, which are small (area $C/\log n$), and essentially form their own Poisson process of intensity $e^{-f(n)}$. (It is very unlikely that two such uncovered regions are close, because the discs bordering them are, on their scale, almost half-planes. Moreover, the expected number of sides of an uncovered region is the same as that of any other "atomic" region, namely 4. To make these heuristics rigorous, one can use the Stein-Chen method [2].)

2.4 Colouring

Both Hall [35] and Janson [38] considered not only the case of coverage, but also that of k-coverage. Our square S_n is said to be k-covered by the discs $D_r(p)$ if every point of S_n is contained in at least k discs. This property is useful for sensor networks, since it allows for the possibility that up to $k-1$ sensors in a small region might simultaneously fail. Now, in our model, for a fixed instance of $\mathcal{P}$, suppose that we increase r until S_n is covered. It turns out that just a small additional increase in r ensures k-coverage.

Theorem 6 ([11, 38]). *For any fixed $k \geq 1$, if*

$$\pi r^2 = \log n + k\log\log n + x,$$

then as $n \to \infty$

$$\mathbb{P}(S_n \text{ is } k\text{-covered}) \to e^{-e^{-x}/(k-1)!}.$$

However, suppose we are more optimistic and instead request the following. We would like to devise a rota system so that each sensor can *sleep* for most of the time, for example, to extend battery life. A natural way of doing this would be to colour the set of sensors with k colours, and arrange that only the sensors with colour ℓ are active in the ℓ^{th} time slot. After k time slots have expired, we repeat the process. In order to detect an event occurring anywhere and at any time, it is necessary that the sensors in each colour class themselves form a single cover of S_n. Thus our question becomes: for fixed k, how large should r be to ensure that the sensors can be partitioned into k groups, each of which covers the sensing region? We call this the problem of *sentry selection*, since each of the groups is a group of sentries keeping watch over the region while the others are sleeping.

It is important to note that a k-cover of an arbitrary set cannot always be partitioned into k single covers. For instance, let S be the set of all subsets of $A = \{1, 2, \ldots, n\}$ of size k. The n sets $S_i = \{B \in S : i \in B\}$, $1 \le i \le n$, form a k-cover of S which cannot even be partitioned into two single covers if $n \ge 2k-1$. This example shows that a solution to our problem must make some use of its geometric setting. Also, even restricting ourselves to discs of equal radii, it is possible to construct k-covers of the plane that are not $\lceil (2k+2)/3 \rceil$-partitionable. Thus we must also make use of the probabilistic setting.

Let $n, r \in \mathbb{R}$. For $k \in \mathbb{N}$, write E_r^k for the event that the discs $D_r(p)$ form a k-cover of S_n, and F_r^k for the event that they may be partitioned into k single covers of S_n. Balister, Bollobás, Sarkar and Walters [11] proved that most random k-covers are in fact k-partitionable.

Theorem 7. *With notation as above,*

$$\mathbb{P}(E_r^k \setminus F_r^k) \le \frac{c_k}{\log n},$$

for some constant c_k.

They also proved that this is sharp, up to the value of the constant c_k. Two hitting time versions of Theorem 7 are also obtained: if we fix n and slowly increase r, or if we fix r and add points uniformly at random to a given area, then with high probability, k-partitionability occurs as soon as we have k-coverage. In particular, Theorem 6 holds also for k-partitionability.

Let us suppose that $\pi r^2 \ge \log n + (k - \frac{1}{2}) \log\log n$ and attempt to prove Theorem 7. For this range of values of r, a typical point in S_n is covered at least $\log n$ times. Intuitively, in most of S_n, we can simply colour the discs randomly, and the probability of *failure*, that is, of a point $x \in S_n$ not being covered by discs of every colour, will be negligible. Indeed, if the level of coverage is at least $3k \log\log n$ everywhere, we can apply the Lovász local lemma to prove that a suitable colouring exists. However, there will be many regions in S_n which are covered less than $3k \log\log n$ times. Call these *thinly covered regions*. It turns out that, with high probability, such regions occur

in small, well-separated clusters. At the scale of the clusters, the curvature of the discs is negligible, so that they behave like half-planes.

Let us examine one such cluster. We will probably find some *very thinly covered regions*, which are covered less than $3k$ times. These turn out to have a very useful property (with high probability): some set of $k-1$ discs $D_1, \dots, D_{k-1}$ covers *all of them*. This facilitates the following simple deterministic colouring method. Suppose that all the discs are actually half-planes. Remove the D_i, and suppose that we still have a cover of S_n (otherwise, we did not have a k-cover to begin with). By Helly's theorem, we can find three of the half-planes which cover S_n, which we colour with colour k and remove. Now restore D_{k-1} and repeat the procedure, colouring three half-planes with colour $k-1$ before removing them. Because of the property mentioned above, we can repeat the process $k-2$ times, using all the colours, and the level of coverage in the cluster will never drop to zero until we have finished. We do this for every cluster, and, outside the clusters, we complete the colouring using the Lovász local lemma, as before.

The actual proofs require somewhat more detailed estimates than the above sketch suggests. As a by-product, we can identify the principal obstructions to k-partitionability in a k-cover as small non-partitionable k-covered configurations which are covered by $k-2$ common discs. Since these configurations are very small, the curvature of the discs forming them is negligible, so that our obstructions are essentially 2-covers with half-planes which cannot be partitioned into two single covers. It is therefore of interest to classify such configurations. Such a classification is presented in [11].

2.5 Thin strips

Suppose that instead of examining points of a Poisson process $\mathcal{P}$ inside a large square, we instead consider the restriction of $\mathcal{P}$ to a thin strip T. As before, we will join two points of $\mathcal{P}$ at distance less than r. Such a model was suggested by the engineering problem of building an electronic "fence" surrounding a large region. The points of $\mathcal{P}$ are sensors, and the fence consists of a thin strip of sensors bordering the region, which has the ability to detect intruders if there is no continuous path crossing it, no point of which lies within distance $r/2$ of any sensor. Note that this is a different condition from both connectivity of the underlying graph $G_r[T]$ of sensors and coverage of the sensing regions (of radius $r/2$). Indeed, if T is a long thin rectangle, our requirement is weaker than both connectivity of $G_r[T]$ and coverage of T by $\bigcup_{p\in\mathcal{P}\cap T} D_{r/2}(p)$. However, it is not hard to see that the new condition is both necessary and sufficient not only for the ability to detect intruders, but also for the ability to relay information longitudinally across T, assuming that the transmission range of the sensors is r. Informally, if there is a continuous

crossing path γ avoiding all the sensing regions, then the sensors on one side of γ will be unable to communicate with those on the other side.

To fix ideas, let $T_h = \mathbb{R} \times [0, h]$, and construct the infinite graph $G_r[T_h]$. Define a *separating path* to be a continuous simple path in T_h starting at some point on the line $y = h$, ending at some point on the line $y = 0$, and not passing strictly within distance $r/2$ of any point of $\mathcal{P} \cap T_h$. This path would be a feasible path for an intruder to take in order to avoid detection. It also identifies a communication breakdown in the information transmission problem. We wish to estimate the frequency with which these paths occur along T_h, but some care is needed with the definition of when two such paths are essentially the same. To this end, we say that a component of $G_r[T_h]$ is *good* if it contains a vertex strictly within distance $\frac{\sqrt{3}}{2}r$ of the top of T_h, and also a vertex strictly within distance $\frac{\sqrt{3}}{2}r$ of the bottom of T_h. The significance of the factor $\frac{\sqrt{3}}{2}$ is that the good components can be ordered along T_h, since no good component can "jump over" another. Now we may define a break in $G_r[T_h]$ to be a partition of the good components into two classes: those on the left of the break and those on the right. It is not hard to see that any separating path defines a break, and conversely that, given a break, there exists a separating path which separates the components on each side of the break. However, two separating paths γ_1 and γ_2 may correspond to the same break. The point of this definition is that the breaks count separating paths that are essentially different.

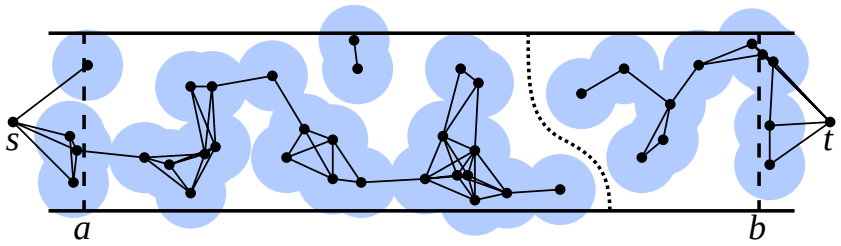

Fig. 1 A break between two good components. Figure taken from [4].

Horizontal translation is an ergodic transformation on the probability space of this model, and consequently it is possible to define the intensity $I_{h,r}$ of breaks along T_h. (In fact, this can also be seen directly.) Loosely speaking, a long section of T_h of length ℓ will contain approximately $\ell I_{h,r}$ breaks. Our problems thus reduce to the single problem of estimating $I_{h,r}$. This was done by Balister, Bollobás, Kumar and Sarkar [4, 6].

Theorem 8. *The intensity of breaks $I(h, r)$ is given by*

$$I_{h,r} = r^{1/3}\varepsilon(hr^{-1/3}) \exp(-hr + O(hr^{-5/3})),$$

where

$$\log \varepsilon(z) = \alpha z + \beta + o_z(1).$$

Numerical simulations give $\alpha \approx 1.12794$ and $\beta \approx -1.05116$. The proof is long and complicated, so we shall content ourselves with a very brief outline. First, a discretization argument shows that, for moderately large values of r and h, the breaks are typically *narrow*, that is, they tend to cut straight across T_h and have width $\Theta(r)$. Next, a version of Theorem 8 is obtained for $h \leq \frac{\sqrt{3}}{2}r$. This involves, among other things, an area-preserving rescaling of T_h near a break, which approximates disc boundaries by parabolas and enables us to replace the two parameters r and h by the single parameter $z = hr^{-1/3}$. (We also make use of a quantitative version of Perron's theorem for eigenvalues of strictly positive matrices.) To extend this to larger values of h, we require two additional lemmas. The first is a technical lemma on the typical *shape* of a break: loosely speaking we require that most breaks are "rectangular". The second lemma states that $I_{h,r}$ is approximately multiplicative, in the sense that if $r \geq 6$ and $h = h_1 + h_2$ with $h_1, h_2 \geq r$, then there is some $c > 0$ such that

$$cr^{-1} I_{h_1,r} I_{h_2,r} \leq I_{h,r} \leq 50h I_{h_1,r} I_{h_2,r}.$$

Naturally, the proof of this begins by splitting the strip T_h into two strips T_{h_1} and T_{h_2}, but many difficulties arise at the interface, and we also require our bound on the expected width of a break established earlier.

For many applications, it is useful to know information about the distribution of the breaks, rather than simply their expected number. It is possible to show, using the Stein-Chen method [2], that for $r \geq 6$ and $x > 0$, the probability that $G_r[T_h]$ restricted to $[0, x/I_{h,r}]$ contains exactly k breaks tends to $e^{-x}x^k/k!$ as $h \to \infty$. For this, we need to know that, for large values of h, the good components are typically wide, and this necessitates a somewhat elaborate discretization argument, owing to complications arising from tiles of our discretization intersecting previously examined regions. For details, see [4].

3 The k-nearest neighbour model

Our second model is very similar to the first. As before, we begin with a Poisson process $\mathcal{P}$ of intensity one in the plane $\mathbb{R}^2$. This time, however, we join each point $p \in \mathcal{P}$ to its k *nearest neighbours*: those points of $\mathcal{P}$ which are the closest, in the usual euclidean norm, to p. (With probability one, there are no ties.) Initially, this creates a directed graph with out-degree k: however, we convert this into an undirected graph by removing the orientations. Note that while the maximum degree of the resulting graph G_k^{nn} may be significantly more than k, the average degree is certainly between k and $2k$, and it is not

hard to see (from elementary properties of the Poisson distribution) that, as $k \to \infty$, the average degree is $(1 + o(1))k$. The maximum degree is at most $6k$ [44].

This is also a very natural model for a transceiver network: one can imagine, for instance, that each transceiver can initiate a connection with at most k others. Indeed, this was the original application, and as such was studied in a series of papers in the engineering literature (see [62] and the references therein).

3.1 Percolation

Percolation in this model is defined as for the Gilbert model, one difference being that k is an integer, so that there is some hope of determining the percolation threshold exactly. To be precise, suppose that the origin is one of the points of $\mathcal{P}$, write $\theta^{\mathrm{nn}}(k)$ for the probability that the origin belongs to an infinite connected component of G_k^{nn}, and define k_c by the formula

$$k_c = \min\{k : \theta(k) > 0\}.$$

Simulations [7] suggest that $\theta^{\mathrm{nn}}(1) = \theta^{\mathrm{nn}}(2) = 0$ and that $\theta^{nn}(3) \approx 0.985$, so that $k_c = 3$, but proving this is another matter. The best published bounds are due to Teng and Yao [53], and Bagchi and Bansal [3], who show that

$$2 \leq k_c \leq 188,$$

although in a paper to be published, Balister, Bollobás and Walters [15] used a certain oriented 1-independent percolation model to prove that

$$k_c \leq 11,$$

and that $k_c = 3$ with confidence 99.99%.

As for the Gilbert model, we can consider the same problem in d dimensions. This was done by Häggström and Meester [34]. Writing $k_c(d)$ for the d-dimensional analogue of k_c, they proved that there exists a d_0 such that

$$k_c(d) = 2 \text{ for all } d \geq d_0,$$

and carried out Monte Carlo simulations which suggest that

$$k_c(d) = \begin{cases} 3 & \text{for } d = 2 \\ 2 & \text{for } d \geq 3. \end{cases}$$

3.2 Connectivity

Since all transceiver networks are finite, it is natural to consider finite versions of the model $G_k^{\rm nn}$. With this in mind, we restrict attention to points of $\mathcal{P}$ within a fixed square S_n of area n, and ask questions about the graph $G_{n,k}$ formed by joining each point of $\mathcal{P}$ within S_n to its k nearest neighbours *within* S_n. Note that this is different from the subgraph of $G_k^{\rm nn}$ induced by the vertices of $\mathcal{P}$ within S_n. One can now ask for an analogue of Penrose's theorem. In other words, how large should we make $k = k(n)$ so as to make $G_{n,k}$ connected with high probability? The obstructions to connectivity cannot be isolated vertices, since there are no isolated vertices in our new model: the minimum degree of $G_{n,k}$ is at least k. On the other hand, it is not hard to see that, for connectivity, we should look at the range $k = \Theta(\log n)$. To see this, imagine tessellating the square S_n with small squares Q_i of area about $\log n$. Then the probability that a small square contains no points of the process is about $e^{-\log n} = n^{-1}$, so that, with high probability, every small square contains at least one point. A short calculation now shows that, if $k \geq 50 \log n$, then $\mathbb{P}(\mathrm{Po}(5\pi \log n) > k) = o(n^{-1})$, so that, again with high probability, every point of $G_{n,k}$ contained in a square Q_i is joined to every other point in Q_i, and also to every point in every adjacent square. This is enough to make $G_{n,k}$ connected. For a lower bound, imagine a small cluster of $k+1$ points surrounded by a large annulus containing no points of $\mathcal{P}$. These points will form a component of $G_{n,k}$ if the thickness of the annulus is greater than the (euclidean) diameter of the cluster it encloses, and if each point outside the annulus has all its k nearest neighbours outside the annulus. It is easy to exhibit an example of such a configuration which occurs in a specified location with probability e^{-ck}: the constant c depends on the exact specifications of the configuration. It is now a simple matter to show that if $e^{-ck} \geq n^{-c'}$ for some $c' < 1$ (i.e. if $k \leq c'' \log n$ for some $c'' < 1/c$), such a configuration will, with high probability, occur somewhere in S_n, disconnecting $G_{n,k}$.

Define c_l and c_u by

$$c_l = \sup\{c : \mathbb{P}(G_{n,\lfloor c \log n \rfloor} \text{ is connected}) \to 0\},$$

and

$$c_u = \inf\{c : \mathbb{P}(G_{n,\lfloor c \log n \rfloor} \text{ is connected}) \to 1\}.$$

Xue and Kumar [62] were the first to publish bounds on c_l and c_u: they obtained $c_l \geq 0.074$ and $c_u \leq 5.1774$, although a bound of $c_u \leq 3.8597$ can be read out of earlier work of Gonzáles-Barrios and Quiroz [33]. Subsequently, Wan and Yi [60] showed that $c_u \leq e$ and Xue and Kumar [63] improved their bound to $c_u \leq 1/\log 2$. The best bounds to date are due to Balister, Bollobás, Sarkar and Walters [7], who proved that $c_l \geq 0.3043$ and $c_u \leq 1/\log 7 \approx 0.5139$.

In some sense, the lower bound comes from optimizing the shape (and other characteristics) of the cluster of $k+1$ points alluded to above. The details are far from straightforward, however, and most of the work consists of optimizing the region *outside* the "empty" annulus. For the upper bound in [7], it is important to show first that the obstructions to connectivity are *small* (of area $C \log n$). For this, in turn, one first needs to observe that no two edges belonging to different components to $G_{n,k}$ may cross, and indeed that, for $k = \Theta(\log n)$, any two edges belonging to different components of $G_{n,k}$ are, with high probability, separated by a certain minimum distance (which depends on k). One can then mimic Penrose's discretization argument to prohibit the existence of two large components, with high probability. The remainder of the proof is very different in character and we will not discuss it here.

The natural conjecture that $c_l = c_u = c$ was made in [7] and proved in [10]. More precisely, we have the following theorem.

Theorem 9. *There exists a constant* c_{crit} *such that if* $c < c_{\text{crit}}$ *and* $k = \lfloor c \log n \rfloor$ *then* $\mathbb{P}(G_{n,k}$ *is connected*$) \to 0$ *as* $n \to \infty$*, and if* $c > c_{\text{crit}}$ *and* $k = \lfloor c \log n \rfloor$ *then* $\mathbb{P}(G_{n,k}$ *is connected*$) \to 1$ *as* $n \to \infty$*.*

One of the ideas in the proof of Theorem 9 is that the essentials of a small component in $G_{n,k}$ can be captured "up to ε" by a sufficiently fine discretization (depending on ε but not on k), which can then be scaled for different values of k. The details, however, are complicated. The proof suggests that $c =$ "0.3043" (where "0.3043" refers to the bound on c_l from [7] mentioned above). To some extent, this is backed up by simulations [7].

From the above results, it follows from the theorems in [9] that also $c_l = c_u = c$ for the problem of s-connectivity, for any fixed s. For information on the directed model $D_{n,k}$, related coverage problems, and several conjectures, see the papers [7, 9].

3.3 Sharp thresholds

We have seen that, for the Gilbert model, very precise results are known about the nature of the transition from non-connectivity to connectivity. For the k-nearest neighbour model, the picture is much less clear, since the obstructions to connectivity are only conjectural. Writing

$$p(n,k) = \mathbb{P}(G_{n,k} \text{ is connected}),$$

let us fix n and focus on the case $k \approx c \log n$, where c is the critical constant from the previous section. We would like to know how quickly $p(n,k)$ changes from almost 0 to almost 1 as k increases. Specifically, write

$$k_n(p) = \min\{\, k : p(n,k) \geq p \,\}.$$

It seems very likely that, for any $0 < \varepsilon < 1$, there exists $C(\varepsilon)$ such that, for all sufficiently large n,

$$k_n(1-\varepsilon) < C(\varepsilon) + k_n(\varepsilon). \tag{1}$$

However, this is not known. What *is* known is that, for fixed k, $p(n,k)$ decreases sharply from almost 1 to almost 0 as n increases. (One has to increase n by a multiplicative factor to make $p(n,k)$ go from $1-\varepsilon$ to ε, but that is only to be expected since $k \approx c \log n$.) Ignoring problems at the boundary, the basic idea is that if $p(n,k) = 1-\varepsilon$, say, then we can consider the square $S_{M^2 n}$ as the union of M^2 copies of S_n, each of which contains a *small* disconnecting component with probability about ε. Consequently,

$$p(M^2 n, k) \approx (1-\varepsilon)^{M^2} < \varepsilon,$$

for a suitable multiplier $M = M(\varepsilon)$. In [9], a weak form of (1) is derived from this result via a complicated double-counting argument.

4 Random Tessellations

Random tessellations of $\mathbb{R}^3$ were introduced into the study of rock formations by Delesse [22] 160 years ago, and in recent years they have been used to study a great variety of problems from kinetics to polymers, ecological systems and DNA replication (see, among others, Evans [25], Fanfoni and Tomellini [26], [27], Ramos, Rikvold and Novotny [51], Tomellini, Fanfoni and Volpe [54], [55], and Pacchiarotti, Fanfoni and Tomellini [46]). In this section we shall concentrate on planar tessellations and give a brief review of the results concerning percolation on the two most frequently studied models, the so called *Voronoi* and *Johnson–Mehl* tessellations.

Strictly speaking, it would suffice to discuss the Johnson–Mehl tessellations only, since a Voronoi tessellation is just a special Johnson–Mehl tessellation. Nevertheless, as Voronoi tessellations have been studied for much longer and are much more basic than Johnson–Mehl tessellations, we shall discuss them in a separate subsection.

In fact, first we shall describe a rather general tessellation in $\mathbb{R}^d$, a trivial extension of the one defined by Johnson and Mehl. Suppose that 'particles' (also called 'nucleation centres') arrive at certain times according to some spatial process, which may be deterministic or random. The moment a particle arrives, it starts to grow a 'crystal' at a certain pace, which may be constant or varying, either deterministically, or in a random way, occupying the 'unoccupied' space around it as it grows. Whatever space a particle occupies belongs to that particle or crystal forever.

In this very general model, the crystal of a 'fast' particle may well overtake and surround the crystal formed by an earlier, but 'slow' particle, and the

crystal of a particle may well consist of an infinite number of components. Not surprisingly, such a general model does not seem to be of much use. Needless to say, it is easy to define even more general models of crystals: e.g., we may use different norms rather than the Euclidean.

In the *Johnson–Mehl model*, all particles have the *same constant* speed, so the crystals are simply connected regions and a particle arriving in the crystal of another particle does not even start to form any crystal of its own, so may be ignored. In a *Voronoi tessellation* the particles not only have the same speed, but also arrive *at the same time*.

4.1 Random Voronoi Percolation

Let us start with a slightly different definition of a Voronoi tessellation. Let Z be a set of points in $\mathbb{R}^d$. (In our terminology above, Z is the set of 'particles' or 'nucleation centres' that grow into 'crystals' or 'tiles'.) For $z \in Z$, let V_z be the closed 'cell' consisting of those points of $\mathbb{R}^d$ that are at most as far from z as from any other point of Z. In all cases of interest, Z is taken to be a countable set without accumulation points; also, Z is not too 'lop-sided': its convex hull is the entire space $\mathbb{R}^d$. In particular, each cell V_z is the intersection of finitely many closed half-spaces, so is a convex polyhedron with finitely many faces. Trivially, each V_z is the closure of its interior U_z; also the total boundary of the cells, $\bigcup_{z \neq z' \in Z} V_z \cap V_{z'}$, has measure 0. The tessellation or tiling of $\mathbb{R}^d$ into the 'cells' or 'tiles' V_z is the *Voronoi tessellation* associated with Z. As we have already remarked, these tessellations were first introduced by Delesse [22] to study the formation of rocks; their mathematical study was initiated only a little later by Dirichlet [23] in connection with quadratic forms, and their detailed study was started by Voronoi [59] about sixty years later. Today, in mathematics they tend to be called Voronoi tessellations (or tilings), although occasionally they are named after Dirichlet.

In a *random Voronoi tessellation* the set Z used to define the Voronoi cells is taken to be a homogeneous Poisson process $\mathcal{P}$ on $\mathbb{R}^d$, of intensity 1, say. The choice of the points z ensures that, with probability 1, the tessellation has no 'pathologies' (in fact, is as 'regular' as possible): any two cells V_z, $V_{z'}$ are either disjoint or share a full $(d-1)$-dimensional face, and in every vertex of a cell precisely $d+1$ cells meet.

Having defined the cells V_z associated with the points $z \in \mathcal{P}$, we define a graph $G_{\mathcal{P}}$ with vertex set $\mathcal{P}$ by joining two vertices by an edge if their cells share a $(d-1)$-dimensional face. Now, a *random Voronoi percolation* in $\mathbb{R}^d$ is simply a site percolation on $G_{\mathcal{P}}$, where $G_{\mathcal{P}}$ itself depends on the random set $\mathcal{P}$. To spell it out, let $0 < p < 1$ be a parameter, and assign one of two *states* to each vertex of $G_{\mathcal{P}}$, *open* or *closed*, such that, given $\mathcal{P}$, each vertex is open with probability p, and the state of a vertex is independent from the set of states assigned to the other vertices. As always, our system

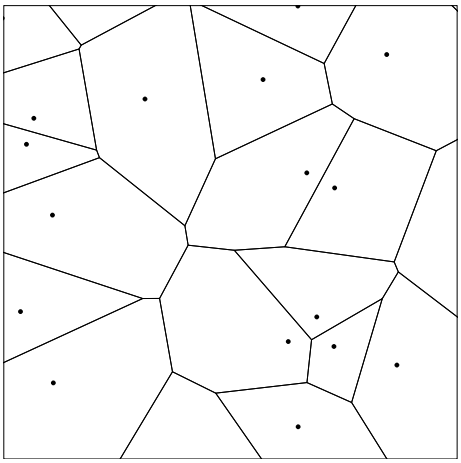

Fig. 2 Part of a random Voronoi tiling in $\mathbb{R}^2$. The dots are the points of a Poisson process. Figure adapted from [19].

is said to *percolate* if the graph $G_{\mathcal{P}}$ contains an infinite path all of whose vertices are open. Equivalently, we may colour a cell *black* with probability p, independently of the colours of the other cells, and colour a point of $\mathbb{R}^d$ black if it belongs to a black cell: then percolation means that the set of black points has an unbounded component.

There is a more user-friendly way of defining random Voronoi percolation: in this approach we take two independent Poisson processes on $\mathbb{R}^d$, $\mathcal{P}^+$ and $\mathcal{P}^-$, with intensities p and $1-p$, respectively. Then $\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^-$ is a Poisson process of intensity 1, $\mathcal{P}^+$ is the set of black (open) points, and $\mathcal{P}^-$ is the set of white (open) points that are used to define Voronoi cells. Define a graph on $\mathcal{P}^+$ by joining two of its points z and z' if there is a path in $\mathbb{R}^d$ from z to z' which does not go nearer to another point of $\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^-$ than to the nearer of z and z'. We have percolation if this graph has an infinite component.

By making use of Kolmogorov's 0-1 law one can show that, for each $0 < p < 1$, the probability of percolation is either 0 or 1. In the first instance, we are interested in the critical probability $p_c = p_c(d)$ such that for $p < p_c$ the probability of percolation is 0, and for $p > p_c$ it is 1.

Unlike in the case of the classical bond and site percolations on lattices, it is not entirely immediate that this critical probability $p_c(d)$ is non-trivial, i.e., $0 < p_c(d) < 1$. A way of showing this is to use $(\mathcal{P}^+, \mathcal{P}^-)$ to define appropriate 1-independent percolations on $\mathbb{Z}^d$ that imply bounds on $p_c(d)$. However, in order to prove better bounds for $p_c(d)$, we have to work rather hard.

For large d, Balister, Bollobás and Quas [5] have proved the following bounds on $p_c(d)$. The proof of the lower bound is fairly easy, but that of the upper bound is more difficult.

Theorem 10. *If d is sufficiently large then the critical probability $p_c(d)$ for random Voronoi percolation on $\mathbb{R}^d$ satisfies*

$$2^{-d}\ (9d\log d)^{-1} \le p_c(d) \le C2^{-d}\sqrt{d}\log d,$$

where C is an absolute constant.

Not surprisingly, most of the interest in random Voronoi percolation centres round percolation in the plane. In fact, in one of the early papers on percolation, Frisch and Hammersley [30] challenged mathematicians to work on problems of this kind. From the late 1970s, much numerical work was done on random Voronoi percolation in the plane (see, e.g., Winterfeld, Scriven and Davis [61], Jerauld, Hatfield, Scriven and Davis [39], and Jerauld, Scriven and Davis [40]). In particular, Winterfeld, Scriven and Davis estimated that the critical probability for random Voronoi percolation in the plane is 0.500 ± 0.010. In spite of this, it was not even *proved* that the critical probability $p_c(2)$ is strictly between 0 and 1.

The 1990s brought about substantial mathematical work on random Voronoi percolation, notably by Vahidi-Asl and Wierman [56, 57, 58], Zvavitch [64], Aizenman [1], Benjamini and Schramm [16] and Freedman [29]. Of these papers, only [64] is about the critical probability: in this paper Zvavitch proved that $p_c(2) \ge 1/2$.

Even without computer experiments, it is difficult not to guess that the critical probability $p_c(2)$ is exactly $1/2$, but a guess like this is very far from a mathematical proof. Such a proof was given by Bollobás and Riordan [19] in 2006.

Theorem 11. *The critical probability for random Voronoi percolation in the plane is* $1/2$.

Very crudely, the 'reason why' the critical probability is $1/2$ is 'self-duality'. For any rectangle R, either there is a 'black crossing' from top to bottom or a 'white crossing' from left to right. In particular, if $p = 1/2$ then the probability that for a given *square* S there is a black crossing from top to bottom is precisely $1/2$. All this is very well, but there are major difficulties in piecing together such crossings to form appropriate paths.

In fact, 'self-duality' is the reason why the critical probability for bond percolation in the plane is $1/2$, but after Harris's proof [37] ten years passed before Kesten [41] could prove the matching upper bound. By now there are numerous elegant and simple proofs of this fundamental Harris–Kesten theorem (see Bollobás and Riordan [20, 18]), but it seems that there is no easy way of adapting any of these proofs to random Voronoi percolation, as the technical problems of overcoming 'singularities' are constantly in the way. Indeed, in order to prove Theorem 11, Bollobás and Riordan [19] had to find a much more involved and delicate argument than those used to tackle percolation on lattices.

To conclude this subsection, let us mention an important question concerning random Voronoi percolation in the plane: is it conformally invariant? (Rather than explaining what this question means, we refer the reader to Benjamini and Schramm [16] and to Chapter 8 of Bollobás and Riordan [18].) Let us just add that in 1994 Aizenman, Langlands, Pouliot and Saint-Aubin [42] made the famous conjecture that under rather weak conditions percolation in the plane is conformally invariant. This has been proved for site percolation in the triangular lattice by Smirnov [52]. Since random Voronoi percolation has much more built-in symmetry than percolation on lattices, like the triangular lattice, it would not be unreasonable to expect that conformal invariance is easiest to prove in this case. Unfortunately, so far this expectation has not been justified.

4.2 Random Johnson–Mehl Percolation

This time we shall consider only Johnson–Mehl percolation in the plane. Let us recall the definition in the simplest case. 'Particles' or 'nucleation centres' arrive randomly on the plane at random times, according to a homogeneous Poisson process $\mathcal{P}$ on $\mathbb{R}^2 \times [0,\infty)$, of intensity 1, say. Thus, if $z = (w,t) \in \mathcal{P}$ then at time t a nucleation centre arrives in the point $w \in \mathbb{R}^2$. As soon as this nucleation centre arrives, it starts to grow at speed 1, say, so that by time $t+u$ it reaches every point x within distance u of w, and claims it for its crystal, provided it had not been claimed by another nucleation centre. A little more formally, if a nucleation centre $w \in \mathbb{R}^2$ arrives at time t then its crystal $V_z = V_{(w,t)}$ consists of all points x such that

$$d_2(x,w) + t \le d_2(x,w') + t'$$

for every point $z' = (w',t') \in \mathcal{P}$. (Here $d_2(x,x')$ is the Euclidean distance of x and x'. In defining a cell, we may safely ignore what happens at the boundary: if a point may be claimed by several particles, we may assign it at random to any one of them.)

In yet another description of this random tessellation, we keep the points $z = (w,t) \in \mathcal{P}$ themselves, grow them in the space $\mathbb{R}^3$ (rather than the plane), and then slice this tessellation with the plane $\mathbb{R}^2 \subset \mathbb{R}^3$. To spell this out, define the Johnson–Mehl norm $||\cdot||_{\mathrm{JM}}$ as the ℓ_1-sum of the ℓ_2-norms on $\mathbb{R}^2$ and $\mathbb{R}$:

$$||(x_1,x_2,t)||_{\mathrm{JM}} = \sqrt{x_1^2+x_2^2} + |t| = ||(x_1,x_2)||_2 + |t|,$$

and write $d = d_{\mathrm{JM}}$ for the corresponding distance. Then the crystal $V_z = V(w,t)$ of the nucleation centre w that arrived at time t is

$$V_z = \{\ x \in \mathbb{R}^2 : d\big((x,0),z\big) = \inf_{z' \in \mathcal{P}} d\big((x,0),z'\big)\ \}. \tag{2}$$

Putting it in this way, we see that Johnson–Mehl tessellations of $\mathbb{R}^2$ correspond to two-dimensional slices of Voronoi tessellations of $\mathbb{R}^3$ with respect to the somewhat unusual sum-metric d_{JM}.

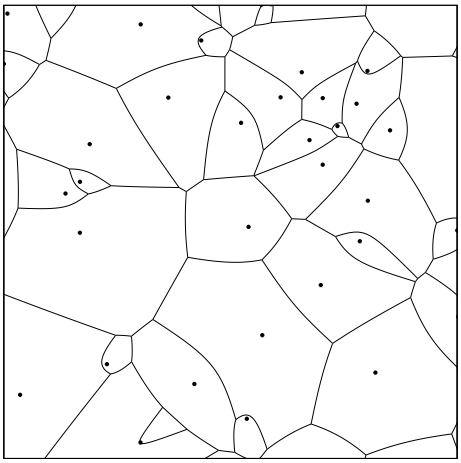

Fig. 3 Part of a random Johnson–Mehl tessellation of $\mathbb{R}^2$. The dots are the projections onto $\mathbb{R}^2$ of those points z of a Poisson process in $\mathbb{R}^2 \times [0,\infty)$ for which the corresponding cell V_z is non-empty. Figure taken from [21].

To define percolation on a random Johnson–Mehl tessellation, we proceed as in the case of Voronoi tessellations: we assign black and white (or open and closed) states to the cells, and look for an unbounded component in the union of black cells.

By adapting their proof of Theorem 11 to the case of random Johnson–Mehl tessellations, Bollobás and Riordan [21] determined the critical probability in this case as well.

Theorem 12. *The critical probability for random Johnson–Mehl percolation in the plane is* $1/2$.

Once again, this result is not too surprising, but what *is* surprising is that although the Johnson–Mehl model is more complicated than the Voronoi model, the proof of this result is actually simpler than that of Theorem 11. This seeming contradiction is explained by the fact that in proving Theorem 12 we can make use of the third dimension in the last representation.

5 Outlook

In this brief review we have seen that although in the past fifty years much work has been done on properties of random geometric graphs, including percolation on them, the subject is still in its infancy. We very much hope that the host of beautiful open problems in the area will attract some beautiful solutions.

References

1. M. Aizenman, Scaling limit for the incipient spanning clusters, in Mathematics of multiscale materials (Minneapolis, MN, 1995–1996), *IMA Vol. Math. Appl.* **99** (1998), 1–24.
2. R. Arratia, L. Goldstein and L. Gordon, Two moments suffice for Poisson approximations: The Chen-Stein method, *Ann. Probab.* **17** (1989), 9–25.
3. A. Bagchi and S. Bansal, On the metric distortion of nearest-neighbour graphs on random point sets, avaliable at http://arxiv.org/abs/0804.3784.
4. P. Balister, B. Bollobás, S. Kumar and A. Sarkar, Reliable density estimates for deployment of sensors in thin strips (detailed proofs), Technical Report, University of Memphis, 2007.
   Available at http://umdrive.memphis.edu/pbalistr/public/ThinStripComplete.pdf
5. P. Balister, B. Bollobás and A. Quas, Percolation in Voronoi tilings, *Random Structures and Algorithms* **26** (2005), 310–318.
6. P. Balister, B. Bollobás, A. Sarkar and S. Kumar, Reliable density estimates for coverage and connectivity in thin strips of finite length, *ACM MobiCom*, Montréal, Canada (2007), 75–86.
7. P. Balister, B. Bollobás, A. Sarkar and M. Walters, Connectivity of random k-nearest neighbour graphs, *Advances in Applied Probability* **37** (2005), 1–24.
8. P. Balister, B. Bollobás, A. Sarkar and M. Walters, Connectivity of a gaussian network, *International Journal of Ad-Hoc and Ubiquitous Computing* **3** (2008), 204–213.
9. P. Balister, B. Bollobás, A. Sarkar and M. Walters, Highly connected random geometric graphs, *Discrete Applied Mathematics* **157** (2009), 309–322.
10. P. Balister, B. Bollobás, A. Sarkar and M. Walters, A critical constant for the k-nearest neighbour model, *Advances in Applied Probability* **41** (2009), 1–12.
11. P. Balister, B. Bollobás, A. Sarkar and M. Walters, Sentry selection in wireless networks, submitted.
12. P. Balister, B. Bollobás and M. Walters, Continuum percolation with steps in an annulus, *Annals of Applied Probability* **14** (2004), 1869–1879.
13. P. Balister, B. Bollobás and M. Walters, Continuum percolation with steps in the square or the disc, *Random Structures and Algorithms* **26** (2005), 392–403.
14. P. Balister, B. Bollobás and M. Walters, Random transceiver networks, to appear in *Advances in Applied Probability*.
15. P. Balister, B. Bollobás and M. Walters, Percolation in the k-nearest neighbour model, submitted.
16. I. Benjamini and O. Schramm, Conformal invariance of Voronoi percolation, *Commun. Math. Phys.* **197** (1998), 75–107.
17. B. Bollobás, Random Graphs, second edition, Cambridge University Press, 2001.
18. B. Bollobás and O.M. Riordan, *Percolation*, Cambridge University Press, 2006, x + 323pp.

19. B. Bollobás and O.M. Riordan, The critical probability for random Voronoi percolation in the plane is 1/2, *Probability Theory and Related Fields* **136** (2006), 417–468.
20. B. Bollobás and O.M. Riordan, A short proof of the Harris–Kesten Theorem, *Bull. London Math. Soc.* **38** (2006), 470–484.
21. B. Bollobás and O.M. Riordan, Percolation on random Johnson–Mehl tessellations and related models, *Probability Theory and Related Fields* **140** (2008), 319–343.
22. A. Delesse, Procédé méchanique pour déterminer la composition des roches, *Ann. des Mines (4th Ser.)* **13** (1848), 379–388.
23. G.L. Dirichlet, Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen, *Journal für die Reine und Angewandte Mathematik* **40** (1850), 209–227.
24. D.W. Etherington, C.K. Hoge and A.J. Parkes, Global surrogates, manuscript, 2003.
25. J.W. Evans, Random and cooperative adsorption, *Rev. Mod. Phys.* **65** (1993), 1281–1329.
26. M. Fanfoni and M. Tomellini, The Johnson–Mehl–Avrami–Kolmogorov model – a brief review, *Nuovo Cimento della Societa Italiana di Fisica. D*, **20** (7-8), 1998, 1171–1182.
27. M. Fanfoni and M. Tomellini, Film growth viewed as stochastic dot processes, *J. Phys.: Condens. Matter* **17** (2005), R571-R605.
28. M. Franceschetti, L. Booth, M. Cook, R. Meester and J. Bruck, Continuum percolation with unreliable and spread-out connections, *Journal of Statistical Physics* **118** (2005), 721–734.
29. M.H. Freedman, Percolation on the projective plane, *Math. Res. Lett.* **4** (1997), 889–894.
30. H.L. Frisch and J.M. Hammersley, Percolation processes and related topics, *J. Soc. Indust. Appl. Math.* **11** (1963), 894–918.
31. E.N. Gilbert, Random plane networks, *J. Soc. Indust. Appl. Math.* **9** (1961), 533–543.
32. E.N. Gilbert, The probability of covering a sphere with N circular caps, *Biometrika* **56** (1965), 323–330.
33. J.M. Gonzáles-Barrios and A.J. Quiroz, A clustering procedure based on the comparison between the k nearest neighbors graph and the minimal spanning tree, *Statistics and Probability Letters* **62** (2003), 23–34.
34. O. Häggström and R. Meester, Nearest neighbor and hard sphere models in continuum percolation, *Random Structures and Algorithms* **9** (1996), 295–315.
35. P. Hall, On the coverage of k-dimensional space by k-dimensional spheres, *Annals of Probability* **13** (1985), 991–1002.
36. P. Hall, On continuum percolation, *Annals of Probability* **13** (1985), 1250–1266.
37. T.E. Harris, A lower bound for the critical probability in a certain percolation process, *Proc. Cam. Philos. Soc.* **56** (1960), 13–20.
38. S. Janson, Random coverings in several dimensions, *Acta Mathematica* **156** (1986), 83–118.
39. G.R. Jerauld, J.C. Hatfield, L.E. Scriven and H.T. Davis, Percolation and conduction on Voronoi and triangular networks: a case study in topological disorder, *J. Physics C: Solid State Physics* **17** (1984), 1519–1529.
40. G.R. Jerauld, L.E. Scriven and H.T. Davis, Percolation and conduction on the 3D Voronoi and regular networks: a second case study in topological disorder, *J. Physics C: Solid State Physics* **17** (1984), 3429–3439.
41. H. Kesten, The critical probability of bond percolation on the square lattice equals 1/2, *Comm. Math. Phys.* **74** (1980), 41–59.
42. R. Langlands, P. Pouliot and Y. Saint-Aubin, Conformal invariance in two-dimensional percolation, *Bull. Amer. Math. Soc.* **30** (1994), 1–61.
43. R. Meester and R. Roy, Continuum Percolation, Cambridge University Press, 1996.
44. G.L. Miller, S.H. Teng and S.A. Vavasis, An unified geometric approach to graph separators, in IEEE 32nd Annual Symposium on Foundations of Computer Science, 1991, 538–547.

45. P.A.P. Moran and S. Fazekas de St Groth, Random circles on a sphere, *Biometrika* **49** (1962), 389–396.
46. B. Pacchiarotti, M. Fanfoni and M. Tomellini, Roughness in the Kolmogorov–Johnson–Mehl–Avrami framework: extension to (2+1)D of the Trofimov–Park model, *Physica A* **358** (2005), 379–392.
47. M.D. Penrose, Continuum percolation and Euclidean minimal spanning trees in high dimensions, *Annals of Applied Probability* **6** (1996), 528–544.
48. M.D. Penrose, The longest edge of the random minimal spanning tree, *Annals of Applied Probability* **7** (1997), 340–361.
49. M.D. Penrose, Random Geometric Graphs, Oxford University Press, 2003.
50. J. Quintanilla, S. Torquato and R.M. Ziff, Efficient measurement of the percolation threshold for fully penetrable discs, *J. Phys. A* **33** (42): L399–L407 (2000).
51. R.A. Ramos, P.A. Rikvold and M.A. Novotny, Test of the Kolmogorov–Johnson–Mehl–Avrami picture of metastable decay in a model with microscopic dynamics *Phys. Rev. B* **59** (1999), 9053–9069.
52. S. Smirnov, Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits, *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, **333** (2001), 239–244.
53. S. Teng and F. Yao, k-nearest-neighbor clustering and percolation theory, *Algorithmica* **49** (2007), 192–211.
54. M. Tomellini, M. Fanfoni and M. Volpe, Spatially correlated nuclei: How the Johnson–Mehl–Avrami–Kolmogorov formula is modified in the case of simultaneous nucleation, *Phys. Rev. B* **62** (2000), 11300–11303.
55. M. Tomellini, M. Fanfoni and M. Volpe, Phase transition kinetics in the case of non-random nucleation, *Phys. Rev. B* **65** (2002), 140301-1 – 140301-4.
56. M.Q. Vahidi-Asl and J.C. Wierman, First-passage percolation on the Voronoi tessellation and Delaunay triangulation, in Random graphs '87 (Poznań, 1987), Wiley, Chichester (1990), pp 341–359.
57. M.Q. Vahidi-Asl and J.C. Wierman, A shape result for first-passage percolation on the Voronoi tessellation and Delaunay triangulation, in *Random graphs, Vol. 2 (Poznań, 1989)*, Wiley-Intersci. Publ., Wiley, New York (1992), pp 247–262.
58. M.Q. Vahidi-Asl and J.C. Wierman, Upper and lower bounds for the route length of first-passage percolation in Voronoi tessellations, *Bull. Iranian Math. Soc.* **19** (1993), 15–28.
59. G. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques, *Journal für die Reine und Angewandte Mathematik*, **133** (1908), 97–178.
60. P. Wan and C.W. Yi, Asymtotic critical transmission radius and critical neighbor for k-connectivity in wireless ad hoc networks, *ACM MobiHoc*, Roppongi, Japan (2004).
61. P.H. Winterfeld, L.E. Scriven and H.T. Davis, Percolation and conductivity of random two-dimensional composites, *J. Physics C* **14** (1981), 2361–2376.
62. F. Xue and P.R. Kumar, The number of neighbors needed for connectivity of wireless networks, *Wireless Networks* **10** (2004), 169–181.
63. F. Xue and P.R. Kumar, On the theta-coverage and connectivity of large random networks, *IEEE Transactions on Information Theory* **52** (2006), 2289–2399.
64. A. Zvavitch, The critical probability for Voronoi percolation, MSc. thesis, Weizmann Institute of Science (1996).