6 Lower bounds in long networks; average-case analysis

Turning to lower bounds, for Ψ^{ave} we start by giving a reformulation (30) of the interpretation (10) in terms of a Poisson point process on the infinite plane. In (10) we required the distribution μ of the network to be translation invariant; by applying a random rotation Θ (uniform on $(0, 2\pi)$) we may suppose also that μ is isotropic. Recall $L(\mu)$ and $S(\mu)$ denote normalized length and stretch. Consider the number

$$\text{intersect}(\mu) = \text{ mean number of intersections of network edges with}$$
$$\text{the } x\text{-axis per unit length.}$$

There is a general formula (see [10] Chapter 8 for the relevant theory) that for any isotropic translation invariant network,

$$L(\mu) = \tfrac{\pi}{2} \times \text{ intersect}(\mu). \tag{29}$$

So we can rewrite (10) as

$$\Psi^{\mathrm{ave}}(s) = \tfrac{\pi}{2} \times \inf\{L(\mu);\ \mu \text{ is isotropic translation invariant, } S(\mu) \leq 1 + s\}. \tag{30}$$

We will use this formulation to obtain an order of magnitude lower bound for small s. This general method was used in a somewhat different context in [1].

Proposition 12 *$\Psi^{ave}(s) = \Omega(s^{-3/8})$ as $s \downarrow 0$.*

Proof. Given $h > 0$ consider the rate-1 Poisson point process restricted to the infinite strip $(-\infty, \infty) \times [-h, h]$. Consider pairs of such Poisson points, where one point is above the x-axis and the other is below the x-axis, and where the line segment between the two points crosses the x-axis at an angle greater that 45°. That is, consider pairs at positions (x_1, y_1) and (x_2, y_2) related by

$$-h < \min(y_1, y_2) < 0 < \max(y_1, y_2) < h; \quad |x_2 - x_1| < |y_2 - y_1|. \tag{31}$$

Call such a pair *friends*. For each friends pair, a hypothetical straight line segment between them crosses the x-axis at some position χ, and the set of all such "virtual crossing positions" is a stationary point process on the line $(-\infty, \infty)$. For $L > 0$ write

$$N(h, L) = \text{number of virtual crossing positions in } [0, L].$$

Now consider a network with stretch $\leq 1 + s$ over the rate-1 Poisson point process on the plane. The route between two friends must cross the x-axis at some "route-crossing position" χ'; write $\delta(h, s)$ for the maximum possible value of the distance between the route-crossing position χ' and the virtual crossing position χ. It is geometrically clear that this maximum is attained when the friends are at positions $(-h, -h)$ and (h, h), and therefore

$$\delta(h, s) = hg^{-1}(s) \tag{32}$$

where $g^{-1}(\cdot)$ is the inverse function of

$$g(\delta) = \frac{\sqrt{1 + (1 + \delta)^2} + \sqrt{1 + (1 - \delta)^2}}{2\sqrt{2}} - 1$$

for which we calculate

$$g(\delta) \sim \delta^2/8 \text{ as } \delta \downarrow 0. \tag{33}$$

Now choose $L > 0$ and partition the x-axis into blocks of length $L + 2\delta(h, s)$, consisting of a middle interval of length L surrounded by two intervals of length $\delta(h, s)$. If the middle interval contains the virtual crossing position for a pair of friends in the Poisson process, then the block contains the route-crossing position, and it follows that the rate of such route-crossing positions is at least $\mathbb{P}(N(h, L) \geq 1)/(L + 2\delta(h, s))$. We may choose h and L arbitrarily, so appealing to (30) we have

$$\Psi^{\text{ave}}(s) \geq \tfrac{\pi}{2} \sup_{h,L} \frac{\mathbb{P}(N(h, L) \geq 1)}{L + 2\delta(h, s)}. \tag{34}$$

We can lower bound the numerator via the second moment inequality

$$\mathbb{P}(N(h, L) \geq 1) \geq \frac{(\mathbb{E}N(h, L))^2}{\mathbb{E}N^2(h, L)}. \tag{35}$$

It is eaasy to calculate $\mathbb{E}N(h, L)$, as follows. For a point $(0, -y_0)$ consider the set of possible positions of a friend (x, y) with $x > 0$. The constraints are

$$0 < y < h, \quad 0 < x < y_0 + y$$

and the area of this region equals $hy_0 + h^2/2$. It follows easily that the rate of the stationary process of virtual crossing positions equals

$$2 \int_0^h (hy_0 + h^2/2) \, dy_0 = 2h^3.$$

The initial factor 2 arises due to the symmetric possibility $(0, +y_0)$ for the left point. So we have shown

$$\mathbb{E}N(h, L) = 2h^3 L.$$

We will be concerned with the limit regime

$$h \to \infty, \; L \to 0, \; 2h^3 L \to \lambda \tag{36}$$

for arbitrary $0 < \lambda < \infty$. Intuitively we expect that the distribution of $N(h, L)$ converges to Poisson(λ) in this regime, but for our purposes it will suffice to prove the second moment result

$$\mathbb{E}N^2(h, L) \to \lambda^2 + \lambda \text{ in the limit regime (36).} \tag{37}$$

Defering the proof of (37), Proposition 12 can be deduced from the ingredients above. Set

$$h = h(s) = s^{-1/8}, \; L = L(s) = s^{3/8}$$

and consider orders of magnitude as $s \downarrow 0$. The numerator in (34) is $\Omega(1)$ by (35) and (37). And by (32) and (33) we see that $\delta(h, s)$ is order $hs^{1/2} = s^{3/8}$, so the denominator in (34) is order $s^{3/8}$, establishing the Proposition.

Proof of (37). The formula for the second moment is given as (38) below. The term $\mathbb{E}N(h, L)$ arises from individual crossings, and the term $(\mathbb{E}N(h, L))^2$ is the contribution from pairs of virtual crossing positions in $[0, L]$ for which the 4 end-points are all distinct. The integral term is the contribution from the case of two virtual crossing positions in $[0, L]$ with an end-point in common, say at $(x_0, -y_0)$ where $y_0 > 0$. This term involves the region $A(x_0, -y_0)$ containing the possible positions of a friend of $(x_0, -y_0)$ for which the virtual crossing position is in $[0, L]$. Figure 5 shows this region, for a particular value of $(x_0, -y_0)$. The integrand $\frac{1}{2}(\text{area } A(x_0, -y_0))^2$ is the mean (conditioned on a point at $(x_0, -y_0)$) number of pairs of friends for which both virtual crossing positions (from friend to $(x_0, -y_0)$) are in $[0, L]$. This leads to the formula

$$\mathbb{E}N^2(h, L) = \mathbb{E}N(h, L) + (\mathbb{E}N(h, L))^2 + 2 \int\int_B \tfrac{1}{2}(\text{area } A(x_0, -y_0))^2 \, dx_0 dy_0. \tag{38}$$

We integrate over the region B of values for $(x_0, -y_0)$ which are consistent with a virtual crossing position in $[0, L]$. This region B can be decomposed as the union of four regions $B_0, B_1^\ell, B_1^r, B_2$ as shown in Figure 6, wherein we are assuming $h > L/2$.

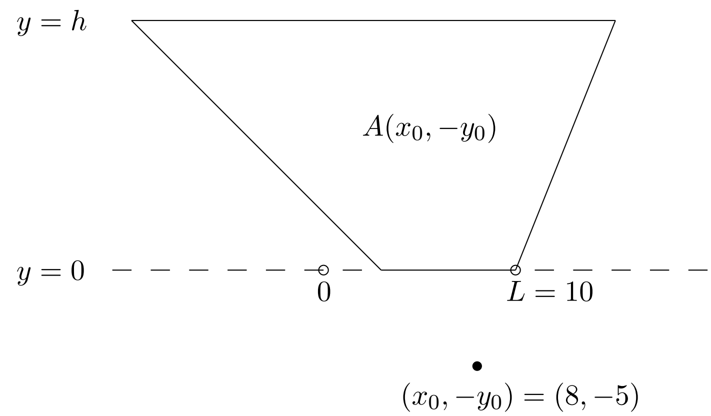

Figure 5. The region $A(x_0, -y_0)$ for the point •.

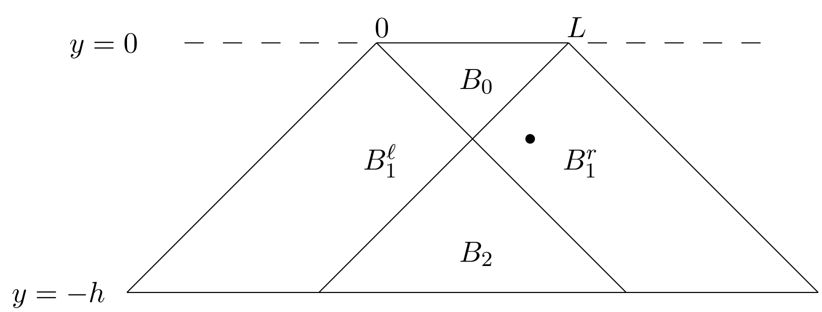

Figure 6. The decomposition of the region of points consistent with a virtual crossing position in $[0, L]$.

For $(x_0, -y_0) \in B_1^r$, the case shown in Figure 5, the region $A(x_0, -y_0)$ is the trapezoid bounded by the line $y = 0$, the line $y = h$, the line of slope -1 through $(x_0, -y_0)$ and the line through $(x_0, -y_0)$ and $(L, 0)$. A brief calculation shows

$$\text{area } A(x_0, -y_0) = \frac{1}{2}\left(1 + \frac{L - x_0}{y_0}\right)\ \left((h + y_0)^2 - y_0^2\right) \qquad \text{for } (x_0, -y_0) \in B_1^r.$$

Easier calculations show

$$\text{area } A(x_0, -y_0) = (h + y_0)^2 - y_0^2 \qquad \text{for } (x_0, -y_0) \in B_0.$$

$$\text{area } A(x_0, -y_0) = \tfrac{L}{2y_0} \ \left((h+y_0)^2 - y_0^2\right) \qquad \text{for } (x_0, -y_0) \in B_2.$$

The case B_1^ℓ is symmetric with B_1^r. We could calculate $\mathbb{E}N^2(h, L)$ exactly using (38), but we only need an upper bound. The formulas above show that, as x_0 varies for fixed y_0, the quantity "area $A(x_0, -y_0)$" takes its maximum value on B_0 or B_2, and so

$$\text{area } A(x_0, -y_0) \leq \left((h+y_0)^2 - y_0^2\right) \ \min(1, \tfrac{L}{2y_0}).$$

So the integral term in (38) is bounded by

$$\int_0^h (L + 2y_0) \ \left(\left((h+y_0)^2 - y_0^2\right) \ \min(1, \tfrac{L}{2y_0})\right)^2 \ dy_0.$$

The integral over $0 < y_0 < L/2$ works out as $\frac{3}{4}h^4L^2 + \frac{5}{6}h^3L^3 + \frac{7}{24}h^2L^4$. The integral over $L/2 < y_0 < h$ works out as $\frac{7}{2}h^4L^2 - \frac{1}{4}h^3L^3 - \frac{3}{4}h^2L^4 + (\frac{1}{2}L^2h^4 + L^3h^3)\log(2h/L)$. So in the limit regime (36), the leading term is the term $\frac{1}{2}L^2h^4 \ \log(2h/L)$. But this term $\to 0$, establishing (37).

7 Lower bounds on Ψ^{worst} based on local optimality

One can get lower bounds on Ψ^{worst} by choosing any configuration of cities and lower bounding the network length required for a network on that particular configuration to have a given stretch. There are heuristic reasons (and the Steiner constant results mentioned at the start of section 3) to suspect that some kinds of regular configurations (rather than typical random configurations) are close to worst-case, so it is not unreasonable to use regular configurations to obtain lower bounds on worst-case behavior. This allows us to work directly on the infinite plane, because the regular configurations we use have known average number of points per unit area.

Consider, for instance, the "square grid" configuration of cities at the points $\{(i,j); -\infty < i,j < \infty\}$. The usual "square lattice" network (roads between city pairs (v,w) at distance 1) has normalized length $= 2$ and stretch $= \sqrt{2}$. It is natural to conjecture this network is optimal, in the following sense.

Conjecture 13 *If a network on the square grid configuration has stretch* $\leq \sqrt{2}$ *then its normalized length is at least* 2.

If true, this would imply $\Psi^{\text{worst}}(\sqrt{2} - 1) \geq 2$. Similarly, any result of the type