

1 STAT 205B homework 7 solutions

Problem 7.4.1 (Durrett).

(a) $EL_1 = \frac{1}{2}$ and $\frac{1}{2}EL_2 = \frac{9}{16}$ by simply counting all the possibilities.

(b) Let $J_n(an)$ be the number of common subsequences of length an when the sequences have total length n . Then

$$\begin{aligned} EJ_n(an) &= \binom{n}{an}^2 2^{-an} = \frac{n!^2 2^{-an}}{(an)!^2 ((1-a)n)!^2} \approx \frac{\left(\frac{n}{e}\right)^{2n} 2\pi n 2^{-an}}{\left(\frac{an}{e}\right)^{2an} 2\pi an \left(\frac{(1-a)n}{e}\right)^{2(1-a)n} 2\pi(1-a)n} \\ &= \frac{2^{-an}}{a^{2an} (1-a)^{2(1-a)n} a 2\pi(1-a)n} \end{aligned}$$

using Stirling's formula, and thus, up to constant order terms in n ,

$$\frac{1}{n} \log EJ_n(an) \approx -a \log 2 - 2a \log a - 2(1-a) \log(1-a).$$

The function on the right is continuous for $a \in (0, 1)$, and has a negative limit as $a \rightarrow 1$. Hence, there exists some $q \in (0, 1)$ (that is, strictly smaller than 1) such that for $a = q$ this function is negative. That implies an exponential decay for $EJ_n(qn)$ in n , therefore $\sum_{n=1}^{\infty} EJ_n(qn) < \infty$. Since $P(J_n(qn) \geq 1) \leq EJ_n(qn)$, the probabilities $P(J_n(qn) \geq 1) = P(\exists \text{ common subsequences of length } qn)$ are also summable. Then the first Borel-Cantelli lemma implies $P(\exists \text{ common subsequences of length } qn \text{ for infinitely many } n) = 0$. Hence,

$$\limsup_{n \rightarrow \infty} \frac{L_{0,n}}{n} \leq q,$$

so $\gamma \leq q < 1$.

Problem 7.5.1 (Durrett).

Let $X_0, Y_0 = 0$. Then $\{X_i - X_{i-1}\}_{i=1}^{\infty}$ are iid, and each coordinate has mean given by:

$$\begin{aligned} EX_1 &= \frac{1}{2}E(X_1 + Y_1) = \frac{1}{2} \int_0^{\infty} P(X_1 + Y_1 > s) ds = \\ &= \frac{1}{2} \int_0^{\infty} P(\nexists \text{ point in the triangle } (0, 0), (0, s), (s, 0)) ds = \frac{1}{2} \int_0^{\infty} e^{-s^2/2} ds = \sqrt{\pi/8} \end{aligned}$$

Thus by the SLLN X_n/n , and similarly Y_n/n , converges to $\sqrt{\pi/8}$. So does M_n/n where $M_n = \lceil X_n \vee Y_n \rceil$. Hence

$$\gamma = \lim Y_{0,n}/n = \lim Y_{0,M_n}/M_n \geq \lim n/M_n = \sqrt{8/\pi}.$$

Problem 7.5.2 (Durrett).

$EJ_k^n = \binom{n}{k}/k!$, that is the number of subsets of size k times the chance that the associated $\pi_n(j)$ form an increasing subsequence. Setting $k = \alpha\sqrt{n}$ and playing with Stirling's formula gives $\log EJ_k^n \sim -2\alpha\sqrt{n}(\log \alpha - 1)$. For $\alpha > e$, the probabilities $P(J_{\alpha\sqrt{n}}^n \geq 1) \leq EJ_{\alpha\sqrt{n}}^n$ are therefore summable, so by Borel-Cantelli $\gamma = \lim Y_{0,n}/n \leq \alpha$. Thus $\gamma \leq e$.

Problem 7.5.3 (Durrett). It is easy to check that Y_n is a Martingale. Hence, $EY_n = EY_1 = 1$. Also, $Y_n \geq (\mu\phi(\theta))^{-n} \exp(-\theta X_{0,n})$. Hence, by Markov inequality

$$P(X_{0,n} \leq an) \leq e^{an\theta} E(Y_n(\mu\phi(\theta))^n) = (\mu\phi(\theta)e^{a\theta})^n \rightarrow 0$$

if $\exp(-\theta a)/\mu\phi(\theta) > 1$.

Problem 7.5.4 (Durrett).

Let M_k be the location of the minimum of $t(m, -k)$, and for $k < \ell$ let $X_{k,\ell}$ be the shortest time it takes to get from the point (M_k, k) to level $-\ell$. Then $\{X_{k,\ell}\}$ satisfies the conditions of Kingman's Theorem, and $\{X_{kn,k(n+1)}\}_{n=0}^\infty$ is iid, hence ergodic. Therefore $a_n/n = X_{0,n}/n$ converges to a constant a.s.