

1 STAT 205B homework 5 solutions

Problem 8.

Let $(Y_n; -\infty < n < \infty)$ be the stationary time-reversed chain. Then the initial position of the cat and the mouse is Y_0 . The mouse jumps to Y_{-1} , and we are interested in

$$T = \min\{n \geq 0 : Y_n = Y_{-1}\}.$$

By stationarity $T \stackrel{d}{=} T^* - 1$ for

$$T^* = \min\{n \geq 1 : Y_n = Y_0\}.$$

But by conditioning on the initial state

$$ET^* = \sum_i \pi(i) E_i T_i^* = \sum_i \pi(i) \frac{1}{\pi(i)} = |S|$$

and so $ET = |S| - 1$.

Problem 9.

(a) The fact that $P(A_{kn})$ depends only on $n - k$ is trivial from stationarity, and using this

$$\begin{aligned} P(X_0 \in B, T_B \geq n) &= P(X_0 \in B, X_1 \notin B, \dots, X_{n-1} \notin B) \\ &= P(A_{1,n-1} \setminus A_{0,n-1}) = P(A_{1,n-1}) - P(A_{0,n-1}) \\ &= P(A_{1,n-1}) - P(A_{1,n}) = P(A_{1,n-1} \setminus A_{1,n}) \\ &= P(X_1 \notin B, \dots, X_{n-1} \notin B, X_n \in B) = P(T_B = n). \end{aligned}$$

(b)

$$\pi(i) E_i T_i = \pi(i) \sum_{n=1}^{\infty} P_i(T_i \geq n) = \sum_{n=1}^{\infty} P(X_0 = i, T_i \geq n) = \sum_{n=1}^{\infty} P(T_i = n) = 1$$

where P denotes the probability measure corresponding to the stationary chain, and we are using the result of part (a) for $B = \{i\}$.

(c)

$$\begin{aligned} \frac{E_i T_i^2}{E_i T_i} &= \pi(i) E_i T_i^2 = \pi(i) \sum_{n=1}^{\infty} (2n - 1) P_i(T_i \geq n) \\ &= \sum_{n=1}^{\infty} (2n - 1) P(X_0 = i, T_i \geq n) = \sum_{n=1}^{\infty} (2n - 1) P(T_i = n) \\ &= 2 \sum_{n=1}^{\infty} n P(T_i = n) - \sum_{n=1}^{\infty} P(T_i = n) = 2ET_i - 1 \\ &= 2 \sum_j \pi(j) E_j T_i - 1 = 2 \sum_j \frac{E_j T_i}{E_j T_j} - 1. \end{aligned}$$

Problem 10. (a) Using the notation $z_{i,j} = \sum_{n=0}^{\infty} (p^n(i,j) - \pi(j))$ from class, and the lemma $\pi(j)E_i T_j = z_{j,j} - z_{i,j}$, we have

$$\begin{aligned} \sum_j \pi(j)E_i T_j &= \sum_j (z_{j,j} - z_{i,j}) = \sum_j z_{j,j} - \sum_j z_{i,j} \\ &= \sum_j z_{j,j} - \sum_j \sum_{n=0}^{\infty} (p^n(i,j) - \pi(j)) = \sum_j z_{j,j} - \sum_{n=0}^{\infty} \sum_j (p^n(i,j) - \pi(j)) \\ &= \sum_j z_{j,j} - \sum_{n=0}^{\infty} (1 - 1) = \sum_j z_{j,j}, \end{aligned}$$

where we used absolute convergence to interchange the sums.

(b) Take the following simple chain on $\{0, 1\}$: $P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$. Then $\pi = (2/3, 1/3)$, $E_0 T_0 = 0$, $E_0 T_1 = 2$, $E_1 T_0 = 1$, $E_1 T_1 = 0$ and

$$\begin{aligned} \sum_i \pi_i E_i T_0 &= \frac{1}{3}, \\ \sum_i \pi_i E_i T_1 &= \frac{4}{3}. \end{aligned}$$

Problem 11.

Note that $g(X_n)$ is a submartingale, and $g(j) < E_j g(X_1)$ for some j . Let $T = \min\{n \geq 1 | X_n = j\}$. Then $g(j) = E_j g(X_T) < E_j g(X_1)$, so $ET = \infty$, which would contradict the optional stopping theorem if $ET < \infty$ held (along with condition (b)). Thus, $ET = \infty$, that is, the chain is not positive recurrent.

An example which is null-recurrent is the simple symmetric random walk on the non-negative integers with $P_{0 \rightarrow 1} = 1$ and with $g(i) = i$. ($j = 0$ in this case.)

Problem 12.

(i)

$$E[\lambda^{-(n+1)} f(X_{n+1}) | X_n = i] = \lambda^{-(n+1)} \sum_j p_{ij} f(j) = \lambda^{-n} f(i)$$

by the assumption, for every state i .

(ii) Let $M_n := \lambda^{-n} f(X_n)$. It suffices to show that for any $\theta > 1/\lambda$ there is at least one state i for which

$$E[\theta^{\tau_b} | X_0 = i] = \infty.$$

Assume to the contrary that all these conditional expectations are finite for some $\theta > 1/\lambda$. then we claim that $f(i) = E_i M_{\tau_b}$.

Since

$$f(i) = E_i M_{\tau_b \wedge n} = f(b)E_i(\lambda^{-\tau_b} \mathbf{1}(\tau_b \leq n)) + E_i(M_n \mathbf{1}(\tau_b > n)).$$

By the assumption that $E_i \theta^{\tau_b} < \infty$, $P_i(\tau_b > n) \leq C\theta^{-n}$. Since the state space is finite, and $\theta > 1/\lambda$, the second term goes to zero as $n \rightarrow \infty$.

Since from assumption it follows that $E_i \lambda^{-\tau_b} < \infty$, by DCT, the first term goes to $f(b)$. This proves the claim that

$$E_i[\lambda^{-\tau_b} f(b)] = f(i). \quad (1)$$

Now, if $f(b) = 0$, then $f(i) = 0$ for every i which contradicts the assumption that f is non-constant. Otherwise, we can assume that $f(b) > 0$. Then (1) implies that $f(i) > 0$ for every i . Let $i^* = \operatorname{argmin} f(i)$. Then

$$f(i^*) \leq \sum_j p_{i^*j} f(j) = \lambda f(i^*)$$

leads to contradiction, since $\lambda < 1$.

Problem 13.

If A_n is the average position of the B particles after n steps, then it is easy to see that

$$\begin{aligned} E[A_{n+1} | \mathbf{X}_n] &= A_n \mathbf{1}\{\text{there is no 0-jump at step } n+1\} + \frac{B}{B-1} A_n \mathbf{1}\{\text{there is a 0-jump at step } n+1\} \\ &= A_n \left(1 + \frac{1}{B-1} \mathbf{1}\{\text{there is a 0-jump at step } n+1\} \right). \end{aligned}$$

This easily implies that $\left(\frac{B}{B-1}\right)^{-N_n} A_n$ is a martingale, where N_n is the number of 0-jumps made up to step n . Using the optional stopping theorem with the stopping time when all the particles reach state K we get

$$E \left(\frac{B}{B-1} \right)^{-N} K = i_0.$$

To get a bound for EN , observe that

$$\frac{i_0}{K} = E \left(1 - \frac{1}{B} \right)^N \geq E \left(1 - \frac{N}{B} \right) = 1 - \frac{EN}{B},$$

which implies

$$EN \geq B(1 - i_0/K).$$