

1 STAT 205B homework 4 solutions

Problem 6.6.6 (Durrett).

Let $T_0 = 0$, and $T_1, T_2, \dots$ be the successive hits to y . Let

$$R_n = \sum_{i=T_n}^{T_{n+1}-1} \mathbf{1}(Y_i = z).$$

Then $R_1, R_2, \dots$ are iid. Let $Z_n = R_0 + \dots + R_n$ so $Z_n/n = R_0/n + (R_1 + \dots + R_n)/n$ converges to the expected number of hits to z between successive visits to y by SLLN. This equals $m(z)/m(y)$. Let $I(n)$ be a random index so that $T_{I(n)} \leq n < T_{I(n)+1}$. Then

$$\frac{Z_{I(n)}}{I(n)} \leq \frac{N_n(z)}{N_n(y)} \leq \frac{Z_{I(n)+1}}{I(n)+1} \cdot \frac{I(n)+1}{I(n)}$$

and both sides converge to $m(z)/m(y)$ a.s.

Problem 6.6.7 (Durrett).

(i) is easy, for (ii) set $b_n = \sum_{k=1}^n \bar{p}_k(y, z)$ and $a_m = p^m(x, y)$. The partial sums $s_n = \sum_{m=1}^n a_m$ diverge. Also, $b_n \uparrow b := m(z)/m(y)$. Using part (i) we can write

$$\frac{\sum_{m=1}^n p^m(x, z)}{\sum_{m=1}^n p^m(x, y)} = \frac{\sum_{m=1}^n \bar{p}_m(x, z)}{s_n} + \frac{\sum_{j=1}^{n-1} a_j b_{n-j}}{s_n}.$$

The first term $\leq (b+1)/s_n \rightarrow 0$, so it suffices to prove that the second term, which we denote by $\bar{b}_n$, converges to b . If we set k_n so that $k_n \rightarrow \infty$ but $(s_n - k_n)/s_n \rightarrow 1$, then

$$b \geq \bar{b}_n \geq \frac{\sum_{j=1}^{n-k_n} a_j b_{n-j}}{s_n} \geq \frac{s_n - k_n}{s_n} b_{k_n} \rightarrow b.$$

Problem 5.

We have to show that $\pi(y) = \sum_{z \in S} \pi(z) p(z, y)$, that is,

$$\sum_{x \in B} \hat{\pi}(x) \mu(x, y) = \sum_{z \in S} \sum_{x \in B} \hat{\pi}(x) \mu(x, z) p(z, y) \quad (1)$$

Using the definition of μ , we have

$$\sum_{z \in S} \mu(x, z) p(z, y) = \sum_{n=0}^{\infty} \sum_{z \in S} P_x(X_n = z, T_B > n) p(z, y) = \sum_{n=0}^{\infty} P_x(X_{n+1} = y, T_B > n)$$

by the Markov property. If $y \notin B$, this equals $\mu(x, y)$ which proves (1). If $y \in B$, then

$$\sum_{n=0}^{\infty} P_x(X_{n+1} = y, T_B > n) = \sum_{n=0}^{\infty} P_x(X_{n+1} = y, T_B = n+1) = P_x(Z_1 = y),$$

so the RHS of (1) is

$$\sum_{x \in B} \hat{\pi}(x) P_x(Z_1 = y) = \hat{\pi}(y)$$

because $\hat{\pi}$ is the stationary distribution of Z . This proves (1) noting that for $y \in B$ $\mu(x, y) = \mathbf{1}\{x = y\}$.

Problem 6.

First, if in some generation the number of people X_k has $\text{Poi}(\lambda')$ distribution, then in the next generation the number of people X_{k+1} is the sum of the number of survivors, which has $\text{Poi}((1-p)\lambda')$ distribution (Poisson thinning), and the number of immigrants (with distribution $\text{Poi}(\lambda)$), so $X_{k+1} \sim \text{Poi}((1-p)\lambda' + \lambda)$. Hence, if we start from $X_0 \sim \text{Poi}(\lambda_0)$, then the distribution of X_n will be Poisson with parameter

$$(1-p)^n \lambda_0 + ((1-p)^{n-1} + (1-p)^{n-2} + \dots + (1-p) + 1)\lambda = (1-p)^n \lambda_0 + \frac{1 - (1-p)^n}{1 - (1-p)} \lambda,$$

which converges to λ/p , so the distribution of X_n converges to the $\text{Poi}(\lambda/p)$ distribution (independently of λ_0).

If we start with an arbitrary distribution for X_0 we can either argue that (X_k) is an irreducible aperiodic Markov chain on the non-negative integers, and we just found that its stationary distribution is $\text{Poi}(\lambda/p)$, or we can deal with the survivors from the first generation and all the rest of the people (new immigrants and survivors among the earlier immigrants) separately. The number of the former tends to 0 a.s. (everyone dies at some point), while the number of the latter is exactly what we calculated above (starting from a $\text{Poi}(\lambda)$ number of first generation immigrants). In either way, we see that the distribution of X_n converges to $\text{Poi}(\lambda/p)$ regardless of the initial distribution.

Problem 7.

(a) Using that $A = [X_m = k \text{ ultimately}]$ is a tail event:

$$\begin{aligned} P(X_{n+1} = i_{n+1} | X_n = i_n, \dots, X_0 = i_0, A) &= \frac{P(X_{n+1} = i_{n+1}, A, X_n = i_n, \dots, X_0 = i_0)}{P(A, X_n = i_n, \dots, X_0 = i_0)} \\ &= \frac{P(X_{n+1} = i_{n+1}, A | X_n = i_n, \dots, X_0 = i_0)}{P(A | X_n = i_n, \dots, X_0 = i_0)} = \frac{P_{i_n}(X_1 = i_{n+1}, A)}{P_{i_n}(A)} \end{aligned}$$

This depends only on i_n , so X_n^* is Markov.

(b)

$$Q(i, j) = \frac{P_i(X_1 = j, A)}{P_i(A)} = p_{ij} \frac{P_j(A)}{P_i(A)}$$

Let $m_i = P_i(A)$. Then $m_0 = 0, m_K = 1$ and

$$m_i = P_{i,i+1} P_{i+1}(A) + P_{i,i-1} P_{i-1}(A) = 2/3 m_{i+1} + 1/3 m_{i-1}, \quad i = 1, 2, \dots, K-1$$

Solving this recursion gives $m_i = \frac{1-(1/2)^i}{1-(1/2)^K}$ for $i = 0, 1, \dots, K$. Hence,

$$Q(i, j) = \begin{cases} \frac{2}{3} \frac{1-(1/2)^j}{1-(1/2)^i} & \text{for } j = i + 1, \ i = 1, 2, \dots, K - 1, \\ \frac{1}{3} \frac{1-(1/2)^i}{1-(1/2)^j} & \text{for } j = i - 1, \ i = 1, 2, \dots, K - 1, \\ 1 & \text{for } j = i = K, \\ 0 & \text{otherwise.} \end{cases}$$

(c)

$$P_{i_0}(\min_{n \geq 0} X_n^* > i) = \frac{P_{i_0}(T_K < T_i)}{P_{i_0}(T_K < T_0)} = \frac{\frac{1-(1/2)^{i_0-i}}{1-(1/2)^{K-i}}}{\frac{1-(1/2)^{i_0}}{1-(1/2)^K}} \text{ for } i = 0, 1, \dots, i_0,$$

$$P_{i_0}(\min_{n \geq 0} X_n^* > i) = 0 \text{ otherwise.}$$