

# 1 STAT 205B homework 3 solutions

**Problem 6.4.2 (Durrett).** (i)

$$\begin{aligned}
 u_n &= P_a(X_n = a) = \sum_{m=1}^n P_a(T_a = m, X_n = a) \\
 &= \sum_{m=1}^n P_a(T_a = m)P_a(X_n = a|T_a = m) \\
 &= \sum_{m=1}^n P_a(T_a = m)P_a(X_n = a|X_m = a) \\
 &= \sum_{m=1}^n P_a(T_a = m)P_a(X_{n-m} = a) \\
 &= \sum_{m=1}^n f_m u_{n-m}.
 \end{aligned}$$

(ii)

$$\begin{aligned}
 u(s)f(s) &= \sum_{n=1}^{\infty} s^n \left( \sum_{m=1}^n f_m u_{n-m} \right) \\
 &= \sum_{n=1}^{\infty} s^n u_n = u(s) - u_0 = u(s) - 1
 \end{aligned}$$

noting that  $f_0 = 0$ . This gives  $u(s) = 1/(1 - f(s))$ . Substituting  $s = 1$  we get  $u(1) = 1 + \sum_{n=1}^{\infty} P_a(X_n = a) = 1 + E_a N(a)$  on the left-hand side, and  $1/(1 - f(1)) = 1/(1 - \rho_{aa})$  on the right-hand side, hence

$$E_a N(a) = \frac{\rho_{aa}}{1 - \rho_{aa}}.$$

**Problem 6.4.3 (Durrett).** (i)

$$(1 - x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x)^n,$$

and substituting  $x = 4pqs^2$  yields

$$\begin{aligned}
 (1 - 4pqs^2)^{-1/2} &= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n (4pqs^2)^n \\
 &= \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} 4^n p^n q^n s^{2n} \\
 &= \sum_{n=0}^{\infty} \binom{2n}{n} p^n q^n s^{2n} = u(s)
 \end{aligned}$$

By the last exercise,  $f(s) = 1 - 1/u(s) = 1 - (1 - 4pqs^2)^{1/2}$ .

(ii) Setting  $s = 1$  gives that the probability of ever returning to 0 is

$$f(1) = 1 - (1 - 4p(1 - p))^{1/2} = 1 - (1 - 4p + 4p^2)^{1/2} = 1 - |1 - 2p|.$$

If  $p \geq 1/2$ , this is  $2 - 2p$ . It can be checked easily that this is the same answer than that of Theorem 5.7.7.

**Problem 6.5.4 (Durrett).**

Let  $N(y)$  be of visits to  $y$  before returning to  $x$  (starting the chain from  $x$ ). Then

$$P_x(N(y) = n) = w_{xy}(1 - w_{yx})^{n-1}w_{yx},$$

hence

$$\mu_x(y) = E_x N(y) = \sum_{n=1}^{\infty} n w_{xy}(1 - w_{yx})^{n-1}w_{yx} = w_{xy}/w_{yx}.$$

**Problem 6.5.5 (Durrett).**

By the uniqueness of the stationary measure up to a constant factor  $\mu_x(\cdot) = c\mu_y(\cdot)$ .

Now  $\mu_y(y) = 1$  gives  $c = \mu_x(y)$ .

**Problem 6.5.8 (Durrett).**

The graph of the possible moves of the knight has the following degrees:

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 4 | 4 | 4 | 3 | 2 |
| 3 | 4 | 6 | 6 | 6 | 6 | 4 | 3 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 4 | 6 | 8 | 8 | 8 | 8 | 6 | 4 |
| 3 | 4 | 6 | 6 | 6 | 6 | 4 | 3 |
| 2 | 3 | 4 | 4 | 4 | 4 | 3 | 2 |

Thus, the stationary distribution of the random walk on the graph is:  $\pi(x) = \frac{\deg(x)}{\sum_y \deg(y)} = \frac{\deg(x)}{336}$  and  $E_{(0,0)}T_{(0,0)} = 1/\pi(0,0) = 168$ .

**Problem 12.**

(a)  $\{T_n = k\} = \{X_0, \dots, X_k \text{ changes values } n \text{ times and } X_k \neq X_{k-1}\} \in \mathcal{F}_k = \sigma(X_0, \dots, X_k)$ , so  $T_0 < T_1 < \dots$  are stopping times. Irreducibility of  $(X_n)$  implies  $P(T_n < \infty) = 1$  and so  $Y_n$  is well-defined and is also a Markov-chain by the strong Markov property. Further  $(Y_n)$  is homogeneous as  $(X_n)$  is and so we can compute the transition matrix of  $(Y_n)$  as

$$\begin{aligned} Q_{ij} &= P(Y_1 = j | Y_0 = i) = P(X_{T_1} = j | X_0 = i) \\ &= \sum_{n=1}^{\infty} P(X_1 = \dots = X_{n-1} = i, X_n = j | X_0 = i) = \sum_{n=1}^{\infty} P_{ij} P_{ii}^{n-1} = \frac{P_{ij}}{1 - P_{ii}}, \quad i \neq j \end{aligned}$$

$$= 0, \quad i = j$$

(b)  $(X_n)$  is recurrent iff some state  $i$  is recurrent  
iff two different states  $i, j$  are recurrent (because of irreducibility)  
iff  $P(X_n = i \text{ i.o. and } X_n = j \text{ i.o.}) = 1$   
iff  $P(Y_n = i \text{ i.o. and } Y_n = j \text{ i.o.}) = 1$  iff  $(Y_n)$  is recurrent.

(c)  $\pi_Q$  is Q-invariant  $\iff \pi_P(i) = \frac{\pi_Q(i)}{1-P_{ii}}$  is P-invariant:  
 $\Rightarrow$ : Fix  $j$ ,  
 $\sum_{i \neq j} \frac{\pi_Q(i)}{1-P_{ii}} P_{ij} + \frac{\pi_Q(j)}{1-P_{jj}} P_{jj} = \pi_Q(j) + \pi_Q(j) \frac{P_{jj}}{1-P_{jj}} = \frac{\pi_Q(j)}{1-P_{jj}}$ .  
 $\Leftarrow$ : Fix  $j$ ,  
 $\sum_{i \neq j} \pi_P(i)(1-P_{ii})Q_{ij} = \sum_{i \neq j} \pi_P(i)P_{ij} = \pi_P(j)(1-P_{jj})$ .

(d) Let  $a_0 = 0, a_1, a_2, \dots > 0, \sum a_i = 1$ . Let

$$((P_{ij})) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdot \\ a_1 & 1-a_1 & 0 & \cdot \\ a_2 & 0 & 1-a_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Then  $P$  is doubly stochastic, so  $\pi = (1, 1, \dots)$  is P-invariant. But  $\sum \pi(i) = \infty$ , so (4.4) and (4.7) imply that  $P$  is null-recurrent.

$\sum \pi(i)(1-P_{ii}) = 2$ , so  $\pi_Q$  may be normalized to a stationary distribution and (4.7) now says that  $Q$  is positive-recurrent.

### Problem 13.

Define  $\pi^*(i) = \pi(i) \sum_{k \in A} p(i, k)$   
Then for  $j \in A$ :

$$\begin{aligned} \sum_{i \in A} \pi^*(i)Q(i, j) &= \sum_{i \in A} \pi(i) \sum_{k \in A} p(i, k) \frac{p(i, j)}{\sum_{k \in A} p(i, k)} \\ &= \sum_{i \in A} \pi(i)p(i, j) = \sum_{i \in A} \pi(j)p(j, i) = \pi^*(j) \end{aligned}$$

Let  $((P_{ij})) = \begin{pmatrix} 0 & .5 & .5 & 0 \\ 0 & 0 & .5 & .5 \\ .5 & 0 & 0 & .5 \\ .5 & .5 & 0 & 0 \end{pmatrix}$  Then,  $\pi(i) = .25$ . Now, let  $A$  be  $\{1, 2, 3\}$ . Then,  $((Q_{ij})) =$

$\begin{pmatrix} 0 & .5 & .5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$  Now,  $\pi^*(1) = \pi^*(3) = .4$  and  $\pi^*(2) = .2$ . However, according to our formula,  $\pi^*(1) = \pi(1) * [p(1, 1) + p(1, 2) + p(1, 3)] = \pi(1) * 1 = .25$ , while  $\pi^*(3) = \pi(3) * [p(3, 1) + p(3, 2) + p(3, 3)] = \pi(3) * .5 = .125$ . Thus, according to our formula,  $\frac{p^*(1)}{p^*(3)}$  should be 2, while it is actually 1. Thus, the formula does not hold.