formula (6.12) implies

$$\begin{aligned} \mathrm{E}(S) &= \int_U \int \frac{y}{\|x\|^2} \lambda(x) p(y)\, dy\, dx \\ &= \int y p(y)\, dy \int_U \frac{1}{\|x\|^2} \lambda(x)\, dx. \end{aligned}$$

Given this naive physical model and a constant $\lambda(x)$, passage to spherical coordinates shows that it is possible for the three-dimensional integral $\int_U \|x\|^{-2} \lambda(x)\, dx$ to diverge on an unbounded set such as $U = \mathsf{R}^3$. The fact that we are not blinded by light on a starlit night suggests that U is bounded. ■

6.10 Problems

1. Consider a Poisson process in the plane with constant intensity λ. Find the distribution and density function of the distance from the origin of the plane to the nearest random point. What is the mean of this distribution? (Hint: Using Example 6.7.1, you should calculate the distribution function $1 - e^{-\lambda\pi r^2}$.)

2. In the context of Example 6.4.2, show that the jth order statistic $X_{(j)}$ had mean and variance

$$\begin{aligned} \mathrm{E}(X_{(j)}) &= \sum_{k=1}^{j} \frac{1}{\lambda(n-k+1)} \\ \mathrm{Var}(X_{(j)}) &= \sum_{k=1}^{j} \frac{1}{\lambda^2(n-k+1)^2}. \end{aligned}$$

3. Continuing Problem 2, prove that $X_{(j)}$ has distribution and density functions

$$\begin{aligned} F_{(j)}(x) &= \sum_{k=j}^{n} \binom{n}{k} (1-e^{-\lambda x})^k e^{-(n-k)\lambda x} \\ f_{(j)}(x) &= n \binom{n-1}{j-1} (1-e^{-\lambda x})^{j-1} e^{-(n-j)\lambda x} \lambda e^{-\lambda x}. \end{aligned}$$

(Hint: Ignore the representation of Example 6.4.2 and reason directly.)

4. In the context of Example 6.4.2, suppose you observe $X_{(1)}, \ldots, X_{(r)}$ and wish to estimate λ^{-1} by a linear combination $S = \sum_{i=1}^{r} \alpha_i X_{(i)}$. Demonstrate that $\mathrm{Var}(S)$ is minimized subject to $\mathrm{E}(S) = \lambda^{-1}$ by taking $\alpha_i = 1/r$ for $1 \le i < r$ and $\alpha_r = (n - r + 1)/r$ [49]

by counting all possible successful sequences of births that lead to either the daughter quota or the son quota being fulfilled first. Combining equations (6.16) and (6.17) permits us to write

$$\mathrm{E}(N_{sd}) \;=\; \frac{s}{p} + \frac{d}{q} - d\sum_{k=0}^{s-1}\binom{d+k}{k}p^k q^d - s\sum_{l=0}^{d-1}\binom{s+l}{l}p^s q^l,$$

replacing a double sum with two single sums.

12. Suppose you randomly drop n balls into m boxes. Assume that a ball is equally likely to land in any box. Use Schrödinger's method to prove that each box receives an even number of balls with probability

$$e_n \;=\; \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}\Big(1-\frac{2j}{m}\Big)^n$$

and an odd number of balls with probability

$$o_n \;=\; \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\Big(1-\frac{2j}{m}\Big)^n.$$

(Hint: The even terms of e^t sum to $\frac{1}{2}(e^t+e^{-t})$ and the odd terms to $\frac{1}{2}(e^t-e^{-t})$.)

13. Continuing Problem 12, show that the probability that exactly j boxes are empty is

$$\binom{m}{j}\sum_{k=1}^{m-j}\binom{m-j}{k}(-1)^{m-j-k}\Big(\frac{k}{m}\Big)^n.$$

14. A one-way highway extends from 0 to ∞. Cars enter at position 0 at times s determined by a Poisson process on $[0,t]$ with constant intensity λ. Each car is independently assigned a velocity v from a density $g(v)$ on $[0,\infty)$. Demonstrate that the number of cars located in the interval (a,b) at time t has a Poisson distribution with mean $\lambda\int_0^t[G(\frac{b}{t-s})-G(\frac{a}{t-s})]\,ds$, where $G(v)$ is the distribution function of $g(v)$ [130].

15. Suppose we generate random circles in the plane by taking their centers (x,y) to be the random points of a Poisson process of constant intensity λ. Each center we independently mark with a radius r sampled from a probability density $g(r)$ on $[0,\infty)$. If we map each random triple (X,Y,R) to the point $U=\sqrt{X^2+Y^2}-R$, then show that the random points so generated constitute a Poisson process with intensity

$$\eta(u) \;=\; 2\pi\lambda\int_0^\infty (r+u)_+ g(r)\,dr.$$

Conclude from this analysis that the number of random circles that overlap the origin is Poisson with mean $\lambda\pi \int_0^\infty r^2 g(r)\,dr$ [143].

16. Continuing Problem 15, perform the same analysis in three dimensions for spheres. Conclude that the number of random spheres that overlap the origin is Poisson with mean $\frac{4\lambda\pi}{3} \int_0^\infty r^3 g(r)\,dr$ [143].

17. If $f(x)$ be a simple function and Π is a Poisson process with intensity function $\lambda(x)$, then demonstrate the formulas in equation (6.13) for the characteristic function and generating function of the random sum S.

18. A train departs at time $t > 0$. During the interval $[0, t]$, passengers arrive at the depot at times T determined by a Poisson process with constant intensity λ. The total waiting time passengers spend at the depot is $W = \sum_T (t - T)$. Show that W has mean $\mathrm{E}(W) = \frac{\lambda t^2}{2}$ and variance $\mathrm{Var}(W) = \frac{\lambda t^3}{3}$ by invoking Campbell's formulas (6.12) and (6.15) [130].

19. Claims arrive at an insurance company at the times T of a Poisson process with constant intensity λ on $[0, \infty)$. Each time a claim arrives, the company pays S dollars, where S is independently drawn from a probability density $g(s)$ on $[0, \infty)$. Because of inflation and the ability of the company to invest premiums, the longer a claim is delayed, the less it costs the company. If a claim is discounted at rate β, then show that the company's ultimate liability $L = \sum_T S e^{-\beta T}$ has mean and variance

$$\begin{aligned}
\mathrm{E}(L) &= \frac{\lambda}{\beta} \int_0^\infty s g(s)\,ds \\
\mathrm{Var}(L) &= \frac{\lambda}{2\beta} \int_0^\infty s^2 g(s)\,ds.
\end{aligned}$$

(Hints: The random pairs (T, S) constitute a marked Poisson process. Use Campbell's formulas (6.12) and (6.15).)