larkov chain ever makes
starts in state i. Show
with j then $f_{ij} = 1$.

of a Markov chain is a
ecurrent and i does not

alk is the Markov chain
gers, and which has the

$= p$

$p = 1/2$, and transient
e chain is called the 1-
walk.
ation, which states that

2π

$a_n/b_n = 1$. You can also
n, then $a_n \sim b_n$ implies
$< \infty$.

andom walk moves on a
transition probabilities

$_{j)} = P_{(i,j),(i,j-1)} = 1/4$

ent.

ymmetric random walk,

rkov chain X_n, let $T_i =$

A

$\min_{i \in A, j \in A} E_i[T_j],$

where $|A|$ denotes the number of elements in A.
(b) Obtain a lower bound for the mean number of flips required until all 2^k patterns of length k have appeared when a fair coin is repeatedly flipped.

7. Consider a Markov chain whose state space is the set of non-negative integers. Suppose its transition probabilities are given by

$$P_{0,i} = p_i, \ i \geq 0, \quad P_{i,i-1} = 1, \ i > 0$$

where $\sum_i ip_i < \infty$. Find the limiting probabilities for this Markov chain.

8. Consider a Markov chain with states $0, 1, \ldots, N$ and transition probabilities

$$P_{0N} = 1, \quad P_{ij} = 1/i, \ i > 0, \ j < i$$

That is, from state 0 the chain always goes to state N, and from state $i > 0$ it is equally likely to go to any lower numbered state. Find the limiting probabilities of this chain.

9. Consider a Markov chain with states $0, 1, \ldots, N$ and transition probabilities

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \ i = 1, \ldots, N-1$$

$$P_{0,0} = P_{N,N} = 1$$

Suppose that $X_0 = i$, where $0 < i < N$. Argue that, with probability 1, the Markov chain eventually enters either state 0 or N. Derive the probability it enters state N before state 0. This is called the *gambler's ruin probability.*

10. If X_n is a stationary ergodic Markov chain, show that $X_1, X_2, \ldots$ is an ergodic sequence.

11. Suppose $X_1, X_2, \ldots$ are iid integer valued random variables with $M_n = \max_{i \leq n} X_i$. Is M_n necessarily a Markov chain? If yes, give its transition probabilities; if no, construct a counterexample.

12. Suppose X_n is a finite state stationary Markov chain, and let $T = \min\{n > 0 : X_n = X_0\}$. Compute $E[T]$.

13. *Hastings-Metropolis algorithm.* Given an irreducible Markov chain with transition probabilities P_{ij} and any positive probability vector $\{\pi_i\}$ for these states, show that the Markov chain with transition probabilities $Q_{ij} = \min(P_{ij}, \pi_j P_{ji}/\pi_i)$ if $i \neq j$ and $Q_{ii} = 1 - \sum_{j \neq i} Q_{ij}$, is time reversible and has stationary distribution $\{\pi_i\}$.

14. Consider a time reversible Markov chain with transition probabilities P_{ij} and stationary probabilities π_i. If A is a set of states of this Markov chain, then we define the A-truncated chain as being a Markov chain whose set of states is A and whose transition probabilities P_{ij}^A, $i, j \in A$, are given by

$$P_{ij}^A = \begin{cases} P_{ij} & \text{if } j \neq i \\ P_{ii} + \sum_{k \notin A} P_{ik} & \text{if } j = i \end{cases}$$

If this truncated chain is irreducible, show that it is time reversible, with stationary probabilities

$$\pi_i^A = \pi_i / \sum_{j \in A} \pi_j, \quad i \in A$$

15. A collection of M balls are distributed among m urns. At each stage one of the balls is randomly selected, taken from whatever urn it is in and then randomly placed in one of the other $m - 1$ urns. Consider the Markov chain whose state at any time is the vector $(n_1, n_2, \ldots, n_m)$ where n_i is the number of balls in urn i. Show that this Markov chain is time reversible and find its stationary probabilities.

16. Let $\mathbf{Q}$ be an irreducible symmetric transition probability matrix on the states $1, \ldots, n$. That is,

$$Q_{ij} = Q_{ji}, \quad i, j = 1, \ldots, n$$

Let $b_i, i = 1, \ldots, n$ be specified positive numbers, and consider

ndom variable with n trials and
pling argument that $\Pr(X \geq k)$
ıd in p for fixed n and k.

with mean λ. Demonstrate that
fixed. (Hint: If $\lambda_1 < \lambda_2$, then
ariables Y_1 and Y_2 with means

es. Between every pair of nodes,
h probability p. If $c(p)$ denotes
ected, then it is intuitively clear
upling proof of this fact.

rs $0, \ldots, m$ with transition prob-

$j = i - 1$
$j = i + 1$

$= 1$. All other transition proba-
ts trapped at 0 or m. Let f_i be
rbed at 0 starting from i. Show
ıe entries of $q = (q_1, \ldots, q_{m-1})$.
or $i = 1, \ldots, m-1$. Construct
and q^* such that $X_0 = Y_0 = i$
X_1. This requires coordinating
Y_1, then run the X_n chain until
case, take another coordinated

total variation norm given in

th success probability p and Y a
ve the total variation inequality

$$\leq \; p^2 \qquad (7.18)$$

of X and Y.

ıriables U_1, U_2, V_1, and V_2 are
ınd V_1 and V_2 are independent.

$$- \pi_{V_1}\| + \|\pi_{U_2} - \pi_{V_2}\|. \qquad (7.19)$$

16. Let $Z_0, Z_1, Z_2, \ldots$ be a realization of a finite-state ergodic chain. If we sample every kth epoch, then show (a) that the sampled chain $Z_0, Z_k, Z_{2k}, \ldots$ is ergodic, (b) that it possesses the same equilibrium distribution as the original chain, and (c) that it is reversible if the original chain is. Thus, based on the ergodic theorem, we can estimate theoretical means by sample averages using only every kth epoch of the original chain.

17. Take three numbers x_1, x_2, and x_3 and form the successive running averages $x_n = (x_{n-3} + x_{n-2} + x_{n-1})/3$ starting with x_4. Prove that

$$\lim_{n\to\infty} x_n \;=\; \frac{x_1 + 2x_2 + 3x_3}{6}.$$

18. Consider a random walk on the integers $\{0, 1, \ldots, n\}$. States 0 and n are absorbing in the sense that $p_{00} = p_{nn} = 1$. If i is a transient state, then the transition probabilities $p_{i,i+1} = p$ and $p_{i,i-1} = q$, where $p + q = 1$. Verify that the hitting probabilities are

$$h_{in} \;=\; \begin{cases} \frac{(\frac{q}{p})^i - 1}{(\frac{q}{p})^n - 1}, & p \neq q \\ \frac{i}{n}, & p = q \end{cases}$$

and the mean hitting times are

$$t_i \;=\; \begin{cases} \frac{n}{p-q}\frac{(\frac{q}{p})^i - 1}{(\frac{q}{p})^n - 1} - \frac{i}{p-q}, & p \neq q \\ i(n-i), & p = q. \end{cases}$$

(Hint: First argue that

$$t_i \;=\; 1 + \sum_{k=1}^{m} p_{ik} t_k$$

in the notation of Section 7.5.)

19. An acceptance function $a : (0, \infty) \mapsto [0, 1]$ satisfies the functional identity $a(x) = x a(1/x)$. Prove that the detailed balance condition

$$\pi_i q_{ij} a_{ij} \;=\; \pi_j q_{ji} a_{ji}$$

holds if the acceptance probability a_{ij} is defined by

$$a_{ij} \;=\; a\Big(\frac{\pi_j q_{ji}}{\pi_i q_{ij}}\Big)$$

in terms of an acceptance function $a(x)$. Check that the Barker function $a(x) = x/(1+x)$ qualifies as an acceptance function and that any acceptance function is dominated by the Metropolis acceptance function in the sense that $a(x) \leq \min\{x, 1\}$ for all x.