8. Prove that

$$E[f(X)] \geq E[f(E[X|Y])] \geq f(E[X])$$

Suppose you want a lower bound on $E[f(X)]$ for a convex function f. The preceding shows that first conditioning on Y and then applying Jensen's inequality to the individual terms $E[f(X)|Y = y]$ results in a larger lower bound than does an immediate application of Jensen's inequality.

9. Let X_i be binary random variables with parameters $p_i, i = 1, \ldots, n$. Let $X = \sum_{i=1}^{n} X_i$, and also let I, independent of the variables $X_1, \ldots, X_n$, be equally likely to be any of the values $1, \ldots, n$. For R independent of I, show that
   (a) $P(I = i|X_I = 1) = p_i/E[X]$
   (b) $E[XR] = E[X]E[R|X_I = 1]$
   (c) $P(X > 0) = E[X]\,E[\frac{1}{X}|X_I = 1]$

10. For X_i and X as in Exercise 9, show that

$$\sum_i \frac{p_i}{E[X|X_i = 1]} \geq \frac{(E[X])^2}{E[X^2]}$$

Thus, for sums of binary variables, the conditional expectation inequality yields a stronger lower bound than does the second moment inequality.
Hint: Make use of the results of Exercises 8 and 9.

11. Let X_i be exponential with mean $8 + 2i$, for $i = 1, 2, 3$. Obtain an upper bound on $E[\max X_i]$, and compare it with the exact result when the X_i are independent.

12. Let $U_i, i = 1, \ldots, n$ be uniform $(0, 1)$ random variables. Obtain an upper bound on $E[\max U_i]$, and compare it with the exact result when the U_i are independent.

13. Let U_1 and U_2 be uniform $(0, 1)$ random variables. Obtain an upper bound on $E[\max(U_1, U_2)]$, and show this maximum is obtained when $U_1 = 1 - U_2$.

11. Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Using inequality (3.7), verify the inequality $n\sqrt[n]{n+1} \leq n + H_n$ for any positive integer n (Putnam Competition 1975).

12. Show that the loglikelihood $L(r)$ in Example 3.4.1 is concave under the reparameterization $r_i = e^{\theta_i}$.

13. Suppose in Example 3.4.2 we minimize the function

$$h_\epsilon(\theta) = \sum_{i=1}^{p} \left\{ \Big[y_i - \sum_{j=1}^{q} x_{ij}\theta_j \Big]^2 + \epsilon \right\}^{1/2}$$

instead of $h(\theta)$ for a small, positive number ϵ. Show that the same MM algorithm applies with revised weights $w_i(\theta^n) = 1/\sqrt{r_i(\theta^n)^2 + \epsilon}$.

14. Suppose the random variables X and Y have densities $f(u)$ and $g(u)$ such that $f(u) \geq g(u)$ for $u \leq a$ and $f(u) \leq g(u)$ for $u > a$. Prove that $\mathrm{E}(X) \leq \mathrm{E}(Y)$. If in addition $f(u) = g(u) = 0$ for $u < 0$, show that $\mathrm{E}(X^n) \leq \mathrm{E}(Y^n)$ for all positive integers n [49].

15. If the random variable X has values in the interval $[a, b]$, then show that $\mathrm{Var}(X) \leq (b-a)^2/4$. (Hints: Reduce to the case $[a,b] = [0,1]$. If $\mathrm{E}(X) = p$, then show that $\mathrm{Var}(X) \leq p(1-p)$.)

16. Let X be a random variable with $\mathrm{E}(X) = 0$ and $\mathrm{E}(X^2) = \sigma^2$. Show that

$$\Pr(X \geq c) \leq \frac{a^2 + \sigma^2}{(a+c)^2} \tag{3.8}$$

for all nonnegative a and c. Prove that the choice $a = \sigma^2/c$ minimizes the right-hand side of (3.8) and that for this choice

$$\Pr(X \geq c) \leq \frac{\sigma^2}{\sigma^2 + c^2}.$$

This is Cantelli's inequality [49].

17. Suppose $g(x)$ is a function such that $g(x) \leq 1$ for all x and $g(x) \leq 0$ for $x \leq c$. Demonstrate the inequality

$$\Pr(X \geq c) \geq \mathrm{E}[g(X)] \tag{3.9}$$

for any random variable X [49]. Verify that the polynomial

$$g(x) = \frac{(x-c)(c+2d-x)}{d^2}$$

with $d > 0$ satisfies t
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then prove that the

Finally, if $\mathrm{E}(X^2) =$

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(c) $B_n f(x) = f(x)$
(d) $B_n x(1-x) =$

20. Suppose the functio
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Conclude from this

21. Let $f(x)$ be a conv
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with $d > 0$ satisfies the stated conditions leading to inequality (3.9). If X is nonnegative with $\mathrm{E}(X) = 1$ and $\mathrm{E}(X^2) = \beta$ and $c \in (0,1)$, then prove that the choice $d = \beta/(1-c)$ yields

$$\Pr(X \geq c) \quad \geq \quad \frac{(1-c)^2}{\beta}.$$

Finally, if $\mathrm{E}(X^2) = 1$ and $\mathrm{E}(X^4) = \beta$, show that

$$\Pr(|X| \geq c) \quad \geq \quad \frac{(1-c^2)^2}{\beta}.$$

18. Let X be a Poisson random variable with mean λ. Demonstrate that the Chernoff bound

$$\Pr(X \geq c) \quad \leq \quad \inf_{t>0} e^{-ct}\,\mathrm{E}(e^{tX})$$

amounts to

$$\Pr(X \geq c) \quad \leq \quad \frac{(\lambda e)^c}{c^c} e^{-\lambda}$$

for any integer $c > \lambda$.

19. Let $B_n f(x) = \mathrm{E}[f(S_n/n)]$ denote the Bernstein polynomial of degree n approximating $f(x)$ as discussed in Example 3.5.1. Prove that

    (a) $B_n f(x)$ is linear in $f(x)$,
    (b) $B_n f(x) \geq 0$ if $f(x) \geq 0$,
    (c) $B_n f(x) = f(x)$ if $f(x)$ is linear,
    (d) $B_n x(1-x) = \frac{n-1}{n} x(1-x)$.

20. Suppose the function $f(x)$ has continuous derivative $f'(x)$. For $\delta > 0$ show that Bernstein's polynomial satisfies the bound

$$\left|\mathrm{E}\left[f\left(\frac{S_n}{n}\right)\right] - f(x)\right| \quad \leq \quad \delta \|f'\|_\infty + \frac{\|f\|_\infty}{2n\delta^2}.$$

Conclude from this estimate that

$$\left\|\mathrm{E}\left[f\left(\frac{S_n}{n}\right)\right] - f\right\|_\infty \quad = \quad O(n^{-\frac{1}{3}}).$$

21. Let $f(x)$ be a convex function on $[0,1]$. Prove that the Bernstein polynomial of degree n approximating $f(x)$ is also convex. (Hint: Show that

$$\begin{aligned}\frac{d^2}{dx^2}\mathrm{E}\left[f\left(\frac{S_n}{n}\right)\right] \quad &= \quad n(n-1)\Big\{\mathrm{E}\left[f\left(\frac{S_{n-2}+2}{n}\right)\right] \\ &\quad -2\,\mathrm{E}\left[f\left(\frac{S_{n-2}+1}{n}\right)\right] + \mathrm{E}\left[f\left(\frac{S_{n-2}}{n}\right)\right]\Big\}.\end{aligned}$$

in the notation of Example 3.5.1.)