9. Suppose $X_1, X_2, \ldots$, are independent and identically distributed mean 0 random variables which each take value +1 with probability 1/2 and take value -1 with probability 1/2. Let $S_n = \sum_{i=1}^{n} X_i$. Which of the following are stopping times? Compute the mean for the ones that are stopping times.
(a) $T_1 = \min\{i \geq 5 : S_i = S_{i-5} + 5\}$
(b) $T_2 = T_1 - 5$
(c) $T_3 = T_2 + 10$.

10. Consider a sequence of independent flips of a coin, and let P_h denote the probability of a head on any toss. Let A be the hypothesis that $P_h = a$ and let B be the hypothesis that $P_h = b$, for given values $0 < a, b < 1$. Let X_i be the outcome of flip i, and set
$$Z_n = \frac{P(X_1, \ldots, X_n|A)}{P(X_1, \ldots, X_n|B)}$$
If $P_h = b$, show that $Z_n, n \geq 1$, is a martingale having mean 1.

11. Let $Z_n, n \geq 0$ be a martingale with $Z_0 = 0$. Show that
$$E[Z_n^2] = \sum_{i=1}^{n} E[(Z_i - Z_{i-1})^2]$$

12. Consider an individual who at each stage, independently of past movements, moves to the right with probability p or to the left with probability $1 - p$. Assuming that $p > 1/2$ find the expected number of stages it takes the person to move i positions to the right from where she started.

13. In Example 3.19 obtain bounds on p when $\theta < 0$.

14. Use Wald's equation to approximate the expected time it takes a random walk to either become as large as a or as small as $-b$, for positive a and b. Give the exact expression if a and b are integers, and at each stage the random walk either moves up 1 with probability p or moves down 1 with probability $1 - p$.

15. Consider a branching process that starts with a single individual. Let π denote the probability this process eventually dies out. With X_n denoting the number of individuals in generation n, argue that π^{X_n}, $n \geq 0$, is a martingale.

16. Given $X_1, X_2, \ldots$, let $S_n = \sum_{i=1}^{n} X_i$ and $\mathcal{F}_n = \sigma(X_1, \ldots X_n)$. Suppose for all n $E|S_n| < \infty$ and $E[S_{n+1}|\mathcal{F}_n] = S_n$. Show $E[X_i X_j] = 0$ if $i \neq j$.

17. Suppose n random points are chosen in a circle having diameter equal to 1, and let X be the length of the shortest path connecting all of them. For $a > 0$, bound $P(X - E[X] \geq a)$.

18. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed discrete random variables, with $P(X_i = j) = p_j$. Obtain bounds on the tail probability of the number of times the pattern $0, 0, 0, 0$ appears in the sequence.

19. Repeat Example 3.27, but now assuming that the X_i are independent but not identically distributed. Let $P_{i,j} = P(X_i = j)$.

20. Let $Z_n, n \geq 0$, be a martingale with mean $Z_0 = 0$, and let $v_j, j \geq 0$, be a sequence of nondecreasing constants with $v_0 = 0$. Prove the *Kolmogorov-Hajek-Renyi inequality*:

$$P(|Z_j| \leq v_j, \text{ for all } j = 1, \ldots, n) \geq 1 - \sum_{j=1}^{n} E[(Z_j - Z_{j-1})^2]/v_j^2$$

21. Consider a gambler who plays at a fair casino. Suppose that the casino does not give any credit, so that the gambler must quit when his fortune is 0. Suppose further that on each bet made at least 1 is either won or lost. Argue that, with probability 1, a gambler who wants to play forever will eventually go broke.

22. What is the implication of the martingale convergence theorem to the scenario of Exercise 10?

23. Three gamblers each start with a, b, and c chips respectively. In each round of a game a gambler is selected uniformly at random to give up a chip, and one of the other gamblers is selected uniformly at random to receive that chip. The game ends when there are only two players remaining with chips. Let X_n, Y_n, and Z_n respectively denote the number of chips the three players have after round n, so $(X_0, Y_0, Z_0) = (a, b, c)$.
(a) Compute $E[X_{n+1} Y_{n+1} Z_{n+1} \mid (X_n, Y_n, Z_n) = (x, y, z)]$.

(b) Show that $M_n = X_n Y_n Z_n + n(a + b + c)/3$ is a martingale.
(c) Use the preceding to compute the expected length of the game.

14. Let $Y_1, \ldots, Y_n$ be independent Bernoulli random variables with success probability μ. Graphically compare the large deviation bound (10.22) to Chebyshev's bound

$$\Pr(|S_n - n\mu| \geq \lambda) \leq \frac{n\mu(1-\mu)}{\lambda^2}$$

when $\mu = 1/2$. Which bound is better? If neither is uniformly better than the other, determine which combinations of values of n and λ favor Chebyshev's bound.

15. Suppose that $v_1, \ldots, v_n \in \mathsf{R}^m$ have Euclidean norms $\|v_i\|_2 \leq 1$. Let $Y_1, \ldots, Y_n$ be independent random variables uniformly distributed on the two-point set $\{-1, 1\}$. If $Z = \|Y_1 v_1 + \cdots + Y_n v_n\|_2$, then prove that

$$\Pr[Z - \mathrm{E}(Z) \geq \lambda\sqrt{n}] \quad \leq \quad e^{-\frac{\lambda^2}{8}}.$$