3.7 Exercises

1. For $\mathcal{F} = \{\phi, \Omega\}$, show that $E[X|\mathcal{F}] = E[X]$.

2. Give the proof of Proposition 3.2 when X and Y are jointly continuous.

3. If $E[|X_i|] < \infty$, $i = 1, \ldots, n$, show that

$$E[\sum_{i=1}^{n} X_i|\mathcal{F}] = \sum_{i=1}^{n} E[X_i|\mathcal{F}]$$

4. Prove that if f is a convex function, then

$$E[f(X)|\mathcal{F}] \geq f(E[X|\mathcal{F}])$$

provided the expectations exist.

5. Let $X_1, X_2, \ldots,$ be independent random variables with mean 1. Show that $Z_n = \prod_{i=1}^{n} X_i$, $n \geq 1$, is a martingale.

6. If $E[X_{n+1}|X_1, \ldots, X_n] = a_n X_n + b_n$ for constants $a_n, b_n, n \geq 0$, find constants A_n, B_n so that $Z_n = A_n X_n + B_n, n \geq 0$, is a martingale with respect to the filtration $\sigma(X_0, \ldots, X_n)$.

7. Consider a population of individuals as it evolves over time, and suppose that, independent of what occurred in prior generations, each individual in generation n independently has j offspring with probability $p_j, j \geq 0$. The offspring of individuals of generation n then make up generation $n+1$. Assume that $m = \sum_j j p_j < \infty$. Let X_n denote the number of individuals in generation n, and define a martingale related to $X_n, n \geq 0$. The process $X_n, n \geq 0$ is called a branching process.

8. Suppose $X_1, X_2, \ldots,$ are independent and identically distributed random variables with mean zero and finite variance σ^2. If T is a stopping time with finite mean, show that

$$\text{Var}(\sum_{i=1}^{T} X_i) = \sigma^2 E(T).$$

9. Suppose $X_1, X_2, ...,$ are independent and identically distributed mean 0 random variables which each take value +1 with probability 1/2 and take value -1 with probability 1/2. Let $S_n = \sum_{i=1}^{n} X_i$. Which of the following are stopping times? Compute the mean for the ones that are stopping times.
(a) $T_1 = \min\{i \geq 5 : S_i = S_{i-5} + 5\}$
(b) $T_2 = T_1 - 5$
(c) $T_3 = T_2 + 10$.

10. Consider a sequence of independent flips of a coin, and let P_h denote the probability of a head on any toss. Let A be the hypothesis that $P_h = a$ and let B be the hypothesis that $P_h = b$, for given values $0 < a, b < 1$. Let X_i be the outcome of flip i, and set
$$Z_n = \frac{P(X_1, \ldots, X_n|A)}{P(X_1, \ldots, X_n|B)}$$
If $P_h = b$, show that $Z_n, n \geq 1$, is a martingale having mean 1.

11. Let $Z_n, n \geq 0$ be a martingale with $Z_0 = 0$. Show that
$$E[Z_n^2] = \sum_{i=1}^{n} E[(Z_i - Z_{i-1})^2]$$

12. Consider an individual who at each stage, independently of past movements, moves to the right with probability p or to the left with probability $1 - p$. Assuming that $p > 1/2$ find the expected number of stages it takes the person to move i positions to the right from where she started.

13. In Example 3.19 obtain bounds on p when $\theta < 0$.

14. Use Wald's equation to approximate the expected time it takes a random walk to either become as large as a or as small as $-b$, for positive a and b. Give the exact expression if a and b are integers, and at each stage the random walk either moves up 1 with probability p or moves down 1 with probability $1 - p$.

15. Consider a branching process that starts with a single individual. Let π denote the probability this process eventually dies out. With X_n denoting the number of individuals in generation n, argue that π^{X_n}, $n \geq 0$, is a martingale.

16. Given $X_1, X_2,$
Suppose for al
$E[X_iX_j] = 0$ i

17. Suppose n ran
eter equal to 1
connecting all

18. Let $X_1, X_2, \ldots$
discrete rando
bounds on the
tern 0, 0, 0, 0

19. Repeat Examp
pendent but n

20. Let $Z_n, n \geq$ (
$v_j, j \geq 0$, be a
Prove the *Kol*

$P(|Z_j| \leq v_j,$ f

21. Consider a gan
casino does no
when his fortu
at least 1 is ei
a gambler wh

22. What is the in
to the scenari

23. Three gamble
In each round
random to gi
selected unifo
ends when th
Let $X_n, Y_n,$ a
the three play
(a) Compute

then the sum $\sum_{i=1}^{n} c_i^2$ figuring in Proposition 10.5.1 can be bounded by

$$\sum_{i=1}^{n} c_i^2 \;\le\; (2\sqrt{2})^2 + 4^2 \sum_{i=1}^{n-1} \frac{1}{n-i} \;\le\; 8 + 16(\ln n + 1).$$

This in turn translates into the Azuma-Hoeffding bound

$$\Pr[|D_n - \mathrm{E}(D_n)| \ge \lambda] \;\le\; 2e^{-\frac{\lambda^2}{48+32\ln n}}.$$

■

10.6 Problems

1. Define the random variables Y_n inductively taking by $Y_0 = 1$ and Y_{n+1} to be uniformly distributed on the interval $(0, Y_n)$. Show that the sequence $X_n = 2^n Y_n$ is a martingale.

2. An urn contains b black balls and w white balls. Each time we randomly withdraw a ball, we replace it by $c+1$ balls of the same color. Let X_n be the fraction of white balls after n draws. Demonstrate that X_n is a martingale.

3. Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables with zero means and common variance σ^2. If $X_n = Y_1 + \cdots + Y_n$, then show that $X_n^2 - n\sigma^2$ is a martingale.

4. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with common moment generating function $M(t) = \mathrm{E}(e^{tY_1})$. Prove that

$$X_n \;=\; M(t)^{-n} e^{t(Y_1+\cdots+Y_n)}$$

is a martingale whenever $M(t) < \infty$.

5. Let Y_n be a finite-state, discrete-time Markov chain with transition matrix $P = (p_{ij})$. If v is a column eigenvector for P with nonzero eigenvalue λ, then verify that $X_n = \lambda^{-n} v_{Y_n}$ is a martingale, where v_{Y_n} is coordinate Y_n of v.

6. Suppose Y_n is the number of particles at the nth generation of a branching process. If s_∞ is the extinction probability, prove that $X_n = s_\infty^{Y_n}$ is a martingale. (Hint: If $Q(s)$ is the progeny generating function, then $Q(s_\infty) = s_\infty$.)

7. In Example 10.3.2, show that $\mathrm{Var}(X_\infty) = \frac{\sigma^2}{\mu(\mu-1)}$ by differentiating equation (10.12) twice. This result is consistent with the mean square convergence displayed in equation (10.9).

8. In Example 10.3.2, show that the fractional linear transformation

$$L_\infty(t) = \frac{pt - p + q}{qt - p + q}$$

solves equation (10.12) when $Q(s) = \frac{p}{1-qs}$ and $\mu = \frac{q}{p}$. Also verify equation (10.13).

9. In Example 10.2.2, suppose that each Y_n is equally likely to assume the values $\frac{1}{2}$ and $\frac{3}{2}$. Show that $\prod_{i=1}^{\infty} Y_i \equiv 0$, but $\prod_{i=1}^{\infty} E(Y_i) = 1$ [19]. (Hint: Apply the strong law of large numbers to the sequence $\ln Y_n$.)

10. Given $X_0 = \mu \in (0,1)$, define X_n inductively by

$$X_{n+1} = \begin{cases} \alpha + \beta X_n, & \text{with probability } X_n \\ \beta X_n, & \text{with probability } 1 - X_n \,. \end{cases}$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Prove that X_n is a martingale with (a) $X_n \in (0,1)$, (b) $\mathrm{E}(X_n) = \mu$, and (c) $\mathrm{Var}(X_n) = [1 - (1-\alpha^2)^n]\mu(1-\mu)$. Also prove that Proposition 10.3.2 implies that $\lim_{n\to\infty} X_n = X_\infty$ exists with $\mathrm{E}(X_\infty) = \mu$ and $\mathrm{Var}(X_\infty) = \mu(1-\mu)$. (Hint: Derive a recurrence relation for $\mathrm{Var}(X_{n+1})$ by conditioning on X_n.)

11. Let $S_n = X_1 + \cdots + X_n$ be a symmetric random walk on the integers $\{-a, \ldots, b\}$ starting at $S_0 = 0$. For the stopping time

$$T = \min\{n : S_n = -a \text{ or } S_n = b\},$$

prove that $\Pr(S_T = b) = a/(a+b)$ by considering the martingale S_n and that $\mathrm{E}(T) = ab$ by considering the martingale $S_n^2 - n$. (Hints: Apply Proposition 10.4.1 and Problem 3.)

12. In the Wright-Fisher model of Example 10.2.6, show that

$$Z_n = \frac{X_n(1 - X_n)}{\left(1 - \frac{1}{2m}\right)^n}$$

is a martingale with values on $[0,1]$. In view of Proposition 10.3.2, $\lim_{n\to\infty} Z_n = Z_\infty$ exists. Thus, $X_n(1-X_n) \approx \left(1 - \frac{1}{2m}\right)^n Z_\infty$ for n large. In other words, X_n approaches either 0 or 1 at rate $1 - \frac{1}{2m}$.

13. In Proposition 10.4.1 suppose we can write either

$$X_n = B_n + I_n \quad \text{or} \quad X_n = B_n - I_n$$

for $n \le T$, where $|B_n| \le c$ and $0 \le I_{n-1} \le I_n$ when $n \le T$. In other words for all times up to T, the B process is bounded and the I process is increasing. Show that $\mathrm{E}(X_T) = \mu$ holds without making assumptions (b) and (c) of the proposition. (Hints: Show that $\mathrm{E}(X_{T\wedge n}) = \mu$ for $T \wedge n = \min\{T, n\}$. Apply the bounded convergence theorem to $B_{T\wedge n}$ and the monotone convergence theorem to $I_{T\wedge n}$.)

14. Let $Y_1, \ldots, Y_n$ be
cess probability μ
(10.22) to Chebys

P

when $\mu = 1/2$. W
than the other, d
favor Chebyshev's

15. Suppose that v_1,
$Y_1, \ldots, Y_n$ be ind
the two-point set
that