3.7 Exercises

1. For $\mathcal{F} = \{\phi, \Omega\}$, show that $E[X|\mathcal{F}] = E[X]$.

2. Give the proof of Proposition 3.2 when X and Y are jointly continuous.

3. If $E[|X_i|] < \infty$, $i = 1, \ldots, n$, show that

$$E[\sum_{i=1}^{n} X_i|\mathcal{F}] = \sum_{i=1}^{n} E[X_i|\mathcal{F}]$$

4. Prove that if f is a convex function, then

$$E[f(X)|\mathcal{F}] \geq f(E[X|\mathcal{F}])$$

provided the expectations exist.

5. Let $X_1, X_2, \ldots,$ be independent random variables with mean 1. Show that $Z_n = \prod_{i=1}^{n} X_i$, $n \geq 1$, is a martingale.

6. If $E[X_{n+1}|X_1, \ldots, X_n] = a_n X_n + b_n$ for constants $a_n, b_n, n \geq 0$, find constants A_n, B_n so that $Z_n = A_n X_n + B_n, n \geq 0$, is a martingale with respect to the filtration $\sigma(X_0, \ldots, X_n)$.

7. Consider a population of individuals as it evolves over time, and suppose that, independent of what occurred in prior generations, each individual in generation n independently has j offspring with probability $p_j, j \geq 0$. The offspring of individuals of generation n then make up generation $n+1$. Assume that $m = \sum_j j p_j < \infty$. Let X_n denote the number of individuals in generation n, and define a martingale related to $X_n, n \geq 0$. The process $X_n, n \geq 0$ is called a branching process.

8. Suppose $X_1, X_2, \ldots$, are independent and identically distributed random variables with mean zero and finite variance σ^2. If T is a stopping time with finite mean, show that

$$\mathrm{Var}(\sum_{i=1}^{T} X_i) = \sigma^2 E(T).$$

then the sum $\sum_{i=1}^{n} c_i^2$ figuring in Proposition 10.5.1 can be bounded by

$$\sum_{i=1}^{n} c_i^2 \;\leq\; (2\sqrt{2})^2 + 4^2 \sum_{i=1}^{n-1} \frac{1}{n-i} \;\leq\; 8 + 16(\ln n + 1).$$

This in turn translates into the Azuma-Hoeffding bound

$$\Pr[|D_n - \mathrm{E}(D_n)| \geq \lambda] \;\leq\; 2e^{-\frac{\lambda^2}{48+32\ln n}}.$$

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10.6 Problems

1. Define the random variables Y_n inductively taking by $Y_0 = 1$ and Y_{n+1} to be uniformly distributed on the interval $(0, Y_n)$. Show that the sequence $X_n = 2^n Y_n$ is a martingale.

2. An urn contains b black balls and w white balls. Each time we randomly withdraw a ball, we replace it by $c+1$ balls of the same color. Let X_n be the fraction of white balls after n draws. Demonstrate that X_n is a martingale.

3. Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables with zero means and common variance σ^2. If $X_n = Y_1 + \cdots + Y_n$, then show that $X_n^2 - n\sigma^2$ is a martingale.

4. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with common moment generating function $M(t) = \mathrm{E}(e^{tY_1})$. Prove that

$$X_n \;=\; M(t)^{-n} e^{t(Y_1+\cdots+Y_n)}$$

is a martingale whenever $M(t) < \infty$.

5. Let Y_n be a finite-state, discrete-time Markov chain with transition matrix $P = (p_{ij})$. If v is a column eigenvector for P with nonzero eigenvalue λ, then verify that $X_n = \lambda^{-n} v_{Y_n}$ is a martingale, where v_{Y_n} is coordinate Y_n of v.

6. Suppose Y_n is the number of particles at the nth generation of a branching process. If s_∞ is the extinction probability, prove that $X_n = s_\infty^{Y_n}$ is a martingale. (Hint: If $Q(s)$ is the progeny generating function, then $Q(s_\infty) = s_\infty$.)

7. In Example 10.3.2, show that $\mathrm{Var}(X_\infty) = \frac{\sigma^2}{\mu(\mu-1)}$ by differentiating equation (10.12) twice. This result is consistent with the mean square convergence displayed in equation (10.9).