12. In a subcritical branching process with immigration, let $Q(s)$ be the progeny generating function and $R(s)$ the generating function of the number of new immigrants at each generation. If the equilibrium distribution has generating function $P_\infty(s)$, then show that

$$P_\infty(s) = P_\infty(Q(s))R(s).$$

For the choices $Q(s) = 1 - p + ps$ and $R(s) = e^{-\lambda(1-s)}$, find $P_\infty(s)$. (Hint: Let $P_\infty(s)$ be a Poisson generating function.)

13. Branching processes can be used to model the formation of polymers [118]. Consider a large batch of identical subunits in solution. Each subunit has $m > 1$ reactive sites that can attach to similar reactive sites on other subunits. For the sake of simplicity, assume that a polymer starts from a fixed ancestral subunit and forms a tree structure with no cross linking of existing subunits. Also assume that each reactive site behaves independently and bonds to another site with probability p. Subunits attached to the ancestral subunit form the first generation of a branching process. Subunits attached to these subunits form the second generation and so forth. In this problem we investigate the possibility that polymers of infinite size form. In this case the solution turns into a gel. Show that the progeny distribution for the first generation is binomial with m trials and success probability p and that the progeny distribution for subsequent generations is binomial with $m-1$ trials and success probability p. Show that the extinction probability t_∞ satisfies

$$\begin{aligned} t_\infty &= (1-p+ps_\infty)^m \\ s_\infty &= (1-p+ps_\infty)^{m-1}, \end{aligned}$$

where s_∞ is the extinction probability for a line of descent emanating from a first-generation subunit. Prove that polymers of infinite size occur if and only if $(m-1)p > 1$.

14. Yeast cells reproduce by budding. Suppose at each generation a yeast cell either dies with probability p, survives without budding with probability q, or survives with budding off a daughter cell with probability r. In the ordinary branching process paradigm, a surviving cell is considered a new cell. If we refuse to take this view, then what is the distribution of the number of daughter cells budded off by a single yeast cell before its death? Show that the extinction probability of a yeast cell line is 1 when $p \geq r$ and $\frac{p}{r}$ when $p < r$ [54].

15. At an X-linked recessive disease locus, there are two alleles, the normal allele (denoted $+$) and the disease allele (denoted $-$). Construct a two-type branching process for carrier females (genotype $+/-$) and affected males (genotype $-$). Calculate the expected numbers f_{ij} of

offspring of each type assuming that carrier females average 2 children, affected males average $2f$ children, all mates are $+/+$ or $+$, and children occur in a 1:1 sex ratio. Note that a branching process model assumes that all children are born simultaneously with the death of a parent. The sensible choice for the death rate λ in a continuous-time model of either type parent is the reciprocal of the generation time, say about $\frac{1}{25}$ per year in humans.

16. In the HIV branching process model, it is of interest to calculate the reproductive potential of a virion in plasma. Because virus reproduction takes place in CD4 cells, let r_i be the expected number of new virions that a particle of type i eventually generates. Show that these numbers obey the following equations:

$$\begin{aligned} r_1 &= \frac{\theta\beta R}{\sigma+\beta R} r_2 + \frac{(1-\theta)\beta R}{\sigma+\beta R} r_3 \\ r_2 &= \frac{\alpha}{\mu+\alpha} r_3 \\ r_3 &= \frac{\delta}{\mu+\delta}\pi. \end{aligned}$$

From these equations calculate

$$r_1 = \frac{\delta\pi\beta R[\alpha+(1-\theta)\mu]}{(\sigma+\beta R)(\mu+\alpha)(\mu+\delta)}.$$

If the reproduction number $r_1 < 1$, then virus numbers keep dropping until extinction. Conversely, virus numbers grow exponentially when $r_1 > 1$. The case $r_1 = 1$ is indeterminate, but a full stochastic analysis of the branching process model demonstrates that extinction is certain in this case as well [12].

17. Consider a multitype branching process with immigration. Suppose that each particle of type i has an exponential lifetime with death intensity λ_i and produces on average f_{ij} particles of type j at the moment of its death. Independently of death and reproduction, immigrants of type i enter the population according to a Poisson process with intensity α_i. If the Poisson immigration processes for different types are independent, then show that the mean number $m_i(t)$ of particles of type i satisfies the differential equation

$$m_i'(t) = \alpha_i + \sum_j m_j(t)\lambda_j(f_{ji} - 1_{\{j=i\}}).$$

Collecting the $m_i(t)$ and α_i into row vectors $m(t)$ and α, respectively, and the $\lambda_j(f_{ji} - 1_{\{j=i\}})$ into a matrix Ω, show that

$$m(t) = m(0)e^{t\Omega} + \alpha\Omega^{-1}(e^{t\Omega} - I),$$

assuming that Ω is invertible. If we replace the constant immigration intensity α_i by the exponentially decreasing immigration intensity $\alpha_i e^{-\mu t}$, then verify that

$$m(t) = m(0)e^{t\Omega} + \alpha(\Omega + \mu I)^{-1}(e^{t\Omega} - e^{-t\mu I}).$$

18. In a certain species, females die with intensity μ and males with intensity ν. All reproduction is through females at an intensity of λ per female. At each birth, the mother bears a daughter with probability p and a son with probability $1-p$. Interpret this model as a two-type, continuous-time branching process with X_t representing the number of females and Y_t representing the number of males, and show that

$$\begin{aligned} \mathrm{E}(X_t) &= \mathrm{E}(X_0)e^{(\lambda p-\mu)t} \\ \mathrm{E}(Y_t) &= \mathrm{E}(X_0)\frac{\lambda(1-p)}{\lambda p+\nu-\mu}e^{(\lambda p-\mu)t} \\ &\quad + \Big[\mathrm{E}(Y_0) - \mathrm{E}(X_0)\frac{\lambda(1-p)}{\lambda p+\nu-\mu}\Big]e^{-\nu t}. \end{aligned}$$

19. In some applications of continuous-time branching processes, it is awkward to model reproduction as occurring simultaneously with death. Birth-death processes offer an attractive alternative. In a birth-death process, a type i particle experiences death at rate μ_i and reproduction of daughter particles of type j at rate β_{ij}. Each reproduction event generates one and only one daughter particle. Thus, in a birth-death process each particle continually buds off daughter particles until it dies. In contrast, each particle of a multitype continuous-time branching process produces a burst of offspring at the moment of its death. This problem considers how we can reconcile these two modes of reproduction. There are two ways of doing this, one exact and one approximate.

    (a) Show that in a birth-death process, a particle of type i produces the count vector $\mathbf{d} = (d_1, \ldots, d_n)$ of daughter particles with probability

    $$p_{i\mathbf{d}} = \frac{\mu_i}{(\mu_i + \beta_i)^{|\mathbf{d}|+1}}\binom{|\mathbf{d}|}{d_1 \ldots d_n}\prod_{k=1}^{n}\beta_{ik}^{d_k},$$

    where $\beta_i = \sum_{j=1}^{n}\beta_{ij}$ and $|\mathbf{d}| = d_1 + \cdots + d_n$. (Hint: Condition on the time of death. The number of daughter particles of a given type produced up to this time follows a Poisson distribution.)

    (b) If we delay all offspring until the moment of death, then we get a branching process approximation to the birth-death process.