Your name: ____

STATISTICS 204 FALL 2007 FINAL

This is a 3-hour in-class exam. There are 7 problems: do as many problems as you can. You are not expected to do them all. Most questions do not require long calculations. Students are allowed a calculator and notes *in their own handwriting* but not other written material.

Write on "question" side of paper only - will be scanned! Extra sheets available. Write full name above, and initials at the ___________ **on top right of each page and any extra sheets.**

1. Let Y have Poisson(μ) distribution. Conditional on $Y = y$ let X have Poisson(y) distribution. Find the probability generating function of $X + Y$.

2. Consider a Poisson point process on the plane with intensity function $\lambda(x_1, x_2) = x_1^2 + x_2^2$. Suppose the points represent trees in a forest.
(a) Let D be the distance from the origin to the closest tree. Find the probability density function of D.
(b) Suppose that a fire burns down each tree with probability $1/3$, independently for different trees. Let D^* be the distance from the origin to the closest tree that is not burned down. Find the probability density function of D^*.

3. Consider a Galton-Watson branching process whose offspring distribution has mean 1 and variance $0 < \sigma^2 < \infty$. Let $(X_n^{(K)},\ n = 0, 1, 2, \ldots)$ be the size of the n'th generation when the intial size is $X_0^{(K)} = K$. Rescale to define

$$Y_{n\delta_K}^{(K)} = K^{-1} X_n^{(K)}$$

that is, regard each generation as living for δ_K of a standard time unit. With appropriate choice of δ_K, for large K the process $Y^{(K)}$ approximates a certain diffusion Y. Find an appropriate definition of δ_K and find the drift rate $\mu(y)$ and variance rate $\sigma^2(y)$ for the diffusion Y.

4. Let (W_t) be standard Brownian motion. Fix $a > 0$ and consider

$$T := \min\{t > 0 : W_t = a \text{ or } = -a\}.$$

We know that $ET = a^2$. Use the martingale

$$W_t^4 - 6tW_t^2 + 3t^2$$

to calculate $E(T^2)$.
[You may assume $E(T^2) < \infty$].

5. A sport team plays n games in a season; suppose each is independently a win with probability p or a loss with probability $1-p$. A *streak of length* k is a sequence of k successive wins which is not part of a sequence of $k+1$ successive wins. Let X be the number of streaks of length k.
(a) Find a simple exact formula for EX.
(b) The distribution of X is approximately Poisson. In major league baseball, $n = 168$ and suppose the best team has $p = 0.65$. Under the model above, what approximately is the probability that this team has a streak of length 11?

6. A professor owns 4 umbrellas, kept at home or the office. Each work day he makes a journey from home to office and a journey from office to home. Suppose on each journey it is raining with probability p (independent for different journeys) and that the professor takes an umbrella if and only if it is raining and an umbrella is available. In the long run, on what proportion of jouneys does the professor get wet (because it is raining but no umbrella is available)?

[Hint: X_n = number of umbrellas at home at end of day n is a Markov chain.]

7. Fix an integer $K \geq 2$. Let $X_1, X_2, \ldots$ be i.i.d. uniform on $\{1, 2, \ldots, K\}$ (cf. the birthday problem). Let

$$T = \min\{n \geq 2 : \; X_n = X_i \text{ for some } 1 \leq i < n\}.$$

Let $a_n = n(n-1)/2$ and let (b_n) be a sequence of constants to be determined later. Define

$$\begin{aligned} M_n &= a_n \text{ if } T > n \\ &= b_T \text{ if } T \leq n. \end{aligned}$$

(a) Calculate $E(M_{n+1} - M_n | X_1, \ldots, X_n)$ on the event $\{T > n\}$.
(b) Find the values for b_n which make the process (M_n) be a martingale.
(c) By applying the optional sampling theorem, find a simple formula for $E(T(T-1))$.