Early Round Bluffing in Poker

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Abstract. Using a simplified form of the Von Neumann and Morgenstern poker calculations, the author explores the effect of hand volatility on bluffing strategy, and shows that one should never bluff in the first round of Texas Hold'Em.

1. INTRODUCTION. The phrase "the mathematics of bluffing" often brings a puzzled response from nonmathematicians. "Isn't that an oxymoron? Bluffing is psychological," they might say, or, "Bluffing doesn't work in online poker. You can't see people's faces." There is a definite psychological aspect to bluffing in poker, as there is in any competitive game. But first and foremost it is a mathematical technique.

The basic mathematics of bluffing in the final betting round has been known for years. We'll recap some of that in this paper. What has not been known previously is how to apply mathematics to bluffing in the early rounds. This paper presents techniques to calculate bluffing strategy for any round of betting in poker. We use these techniques to demonstrate a useful fact—that it is not profitable to bluff in the first round of Texas Hold'Em poker.

2. VON NEUMANN AND MORGENSTERN. The mathematical study of bluffing has its roots in papers published in the early 20th century by Emile Borel and John Von Neumann. The biggest influence came from Von Neumann's article, which he originally published in German [**6**]. He translated the article to English for his book with Oskar Morgenstern, *Theory of Games and Economic Behavior* [**7**], which we will refer to as VNM. In order to talk about poker mathematically, the authors made a number of simplifications.

Their game can be described by the extensive form diagram in Figure 1. At the start of the game, there is one unit in the pot. The first move is a move of Nature, assigning a random number from [0, 1] to each player. In our description, the lower hand is better, that is, zero is the best possible hand. Player 1 (we'll refer to Player 1 as "him" and Player 2 as "her") then makes a decision to Bet or to Check. If he checks, there is an immediate showdown, which is to say that the hands are exposed and compared. Whoever has the lowest hand wins the one-unit pot.

If Player 1 bets, putting one unit into the pot, then there is a move for Player 2. She decides to call or drop. If she drops, then Player 1 gets the pot, +1 for him. If she calls, adding one unit to the pot, there is a showdown with the lowest hand winning the three unit pot. This makes a win of 2 units or a loss of 1 unit, depending on the values of the hands.

VNM showed that this game has a pure strategy solution. (They also show a mixed strategy solution, and multiple pure strategy solutions. We will use only the simplest pure strategy.) The strategy is that Player 2 selects a threshold c, such that she will call whenever her hand is better (lower) than c, and drop if her hand is worse. The Player 1 strategy that we will analyze has two parameters, b (bet) and d (deceive). He will bet if his hand is better than b. He will also bet (that is to say *bluff*) if his hand is worse than d.

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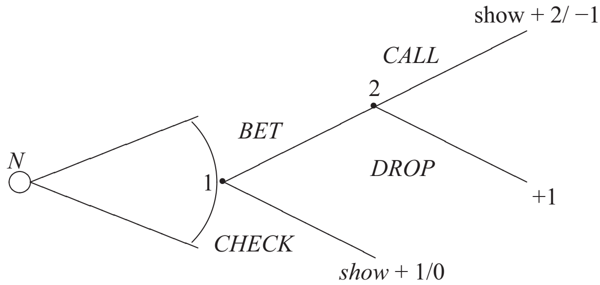

Figure 1. Extensive form of the VNM poker game

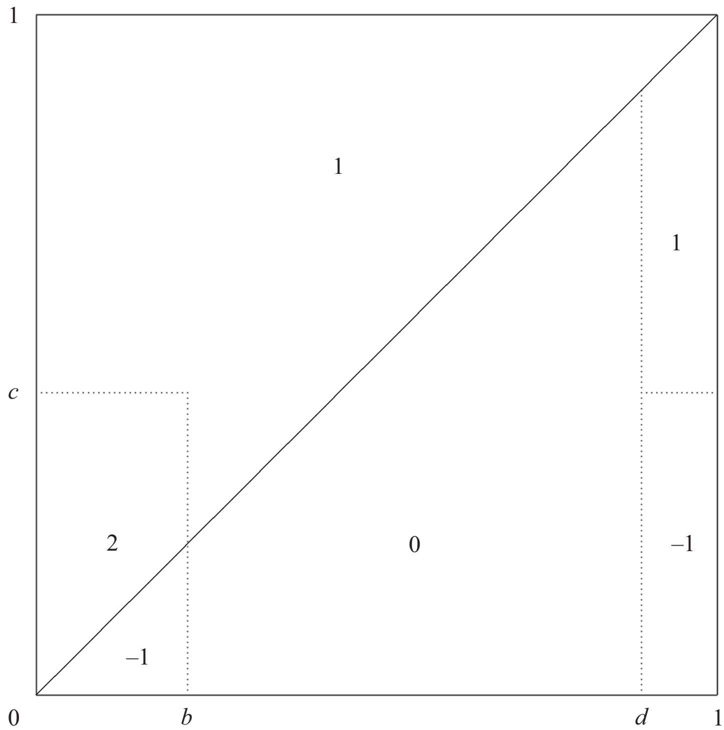

Figure 2. Payoff square for Player 1 in the VNM game

A complete strategy for the game is given by $s_1 = (b, d)$ and $s_2 = (c)$. We want a *Nash Equilibrium*, which you will recall is a strategy in which neither player can benefit from unilaterally changing their own strategy. When we say *equilibrium*, we mean Nash Equilibrium. Following Zhang's example [**9**], in Figure (2) we present a "payoff square" giving the payoffs of the game.

VNM calculated the equilibrium strategy for this game to be $s_1 = (\frac{2}{9}, \frac{8}{9})$, $s_2 = (\frac{4}{9})$. Player 1 always bets with a hand better than $\frac{2}{9}$, and always bluffs with a hand worse than $\frac{8}{9}$. Player 2 always calls with a hand better than $\frac{4}{9}$. (Note that for these calculations, and all the others we discuss, the players behave hyper-rationally. They are aware of all the odds and assume that their opponent is, too.)

The *value* of the game, which is the amount Player 1 can expect to win on average using the equilibrium strategy, is $\frac{5}{9}$.

Note that in this solution $0 < b < c < d < 1$.

Definition 2.1. *A bluff is a bet made with a hand in the range* $[d, 1]$.

Our definition of bluffing involves betting with your worst hands. This does not include aggressive betting with hands almost good enough for a normal bet (see discussion of a semibluff in Section 11). Nor does it include betting with every hand ($b = d$). We say there is no bluffing without the strict inequality $b < d < 1$.

The VNM game may seem very simple, but at the time it was published, it was a major breakthrough. For one thing, it established the bluff as a mathematical strategy. VNM shows that the bluff is a profitable strategy, even if your opponent knows exactly which hands you use for bluffing.

In the equilibrium solution, each player's strategy makes their opponent indifferent to which strategy they choose for their marginal hands. That is to say that the values b, c, and d will satisfy equations (1) below. The function u is the utility, or expected return for a player. It has two arguments, the hand that the player holds, and the action that they take. The equations say that at the margin between betting and checking b, Player 1 gets an equal return whether he bets or checks. The same is true for the margin between checking and bluffing, and the margin between calling and dropping.

Consider the equations

$$\begin{aligned} u_1(b,\ \text{bet}) &= u_1(b,\ \text{check}), \\ u_1(d,\ \text{bet}) &= u_1(d,\ \text{check}), \\ u_2(c,\ \text{call}) &= u_2(c,\ \text{drop}). \end{aligned} \tag{1}$$

A published remark by poker theorist Norman Zadeh [**8**] led us to use such *indifference equations* to calculate poker strategies in a previous paper [**2**]. This technique was refined by Ferguson et al. [**4**], who introduced the term "indifference equation."

3. INDIFFERENCE EQUATIONS. We'd like to justify the use of indifference equations and demonstrate their usefulness in poker calculations. We begin with a graphical, visual presentation of Player 1's expectation in the VNM game using two simple strategies.

In Figure 3 (left), he uses the strategy $b = 1$ (bet with any hand less than 1, which is to say always), and $d = 1$ (never bluff). Player 2 is using the equilibrium strategy discussed in Section 2, which is to call with hands lower than $\frac{4}{9}$, and fold otherwise. When Player 1 uses this all-bet strategy, his expectation for the best possible hand (zero) is two units when Player 2 calls, and one unit when she drops, for an expectation of $2 \cdot \frac{4}{9} + 1 \cdot \frac{5}{9} = \frac{13}{9}$. His expectation decreases linearly for higher hand values, until it reaches Player 2's calling threshold of $\frac{4}{9}$. For hands worse than that, he loses one unit when she calls, and wins one unit when she drops, for a constant expectation of $\frac{1}{9}$.

In Figure 3 (right), Player 1 is using the strategy $b = 0$ and $d = 1$. He never bets. His expectation is just what he gets from a showdown for the 1 unit in the pot.

Figure 4 is a superposition of the other two. It illustrates that for hands between 0 and $\frac{2}{9}$, Player 1 has a higher expectation from betting. For hands between $\frac{2}{9}$ and $\frac{8}{9}$, he has a higher expectation from checking. And for hands between $\frac{8}{9}$ and 1, he has a higher expectation from betting. These values are in fact the ones found by Von Neumann, as discussed in Section 2.

Ferguson et al. [**4**] provide a proof of the validity of indifference equations for some games with no volatility. We cannot prove the general case, but do provide some discussion in Section 7, and make the following Claim.

Claim 3.1. *A solution to indifference equations is a Nash equilibrium.*

Once we accept indifference equations as a tool, poker calculations become much simpler. We can easily calculate the VNM result in just a few lines.

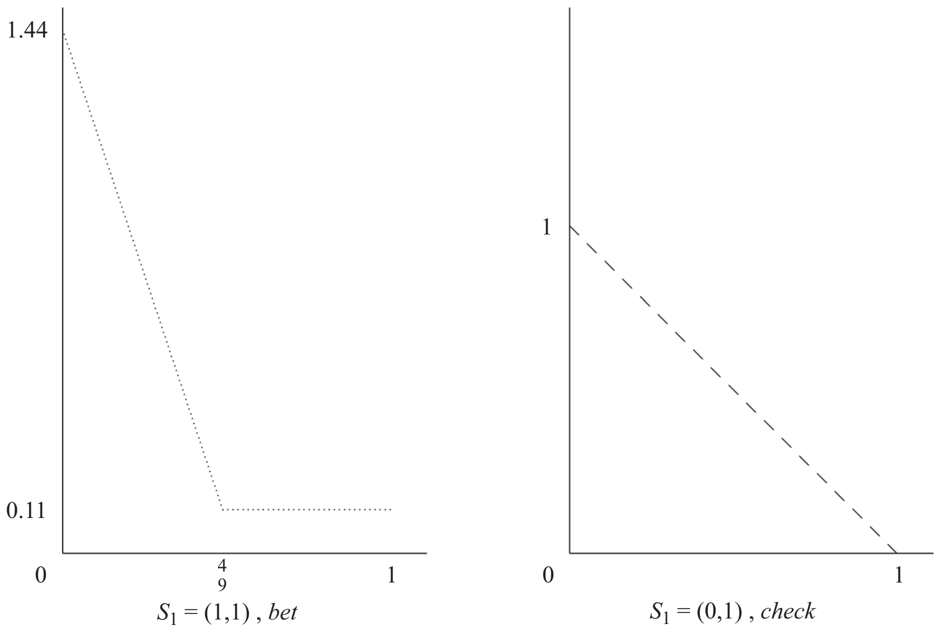

Figure 3. Player 1 pure strategy payoffs vs value of hand

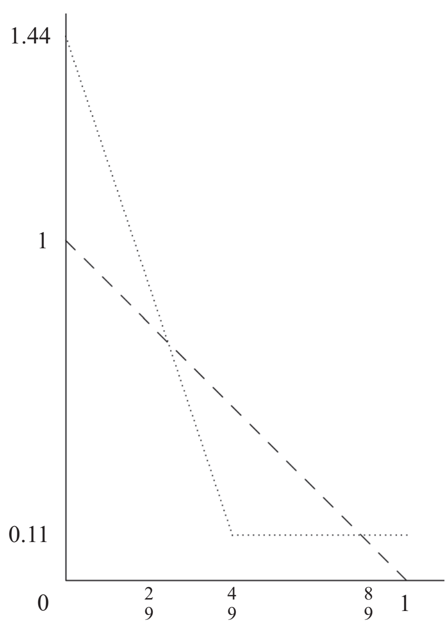

Figure 4. Superposition of simple strategies

The final line of (1) comes into play only if Player 1 has a hand in $[0, b]$ or $[d, 1]$, which causes him to make some kind of bet. The left side of the equation is Player 2's cost (-1) plus her possible win (3) times the conditional probability that Player 1 was bluffing, given that he bet. The right side of the equation is what Player 2 gets by dropping, which is zero. This gives us

$$-1 + \frac{3(1-d)}{(b+1-d)} = 0. \tag{2}$$

This simplifies to $b + 2d = 2$.

Switching to Player 1's indifference equations, the top two lines of (1), we note that if he checks any arbitrary hand h, his expectation is one (the pot size) times the probability that the opponent has a worse hand. Since his opponent's hand is uniformly distributed in $[0, 1]$, that means $u_1(h, \text{ check}) = 1 - h$, where h is Player 1's hand. His betting expectation is the cost of the bet (-1), plus 2 units if Player 2 drops, plus 3 units if she calls with a worse hand,

$$u_1(h, bet) = \begin{cases} -1 + 2(1-c) + 3(c-h) & \text{if } h \le c, \\ -1 + 2(1-c) & \text{if } h > c. \end{cases} \tag{3}$$

Using our knowledge that $b < c < d$, and letting $h = b$ and $h = d$ in the above, we obtain two more equations in b, c, and d,

$$\begin{aligned} -1 + 2(1-c) + 3(c-b) &= 1 - b, \\ -1 + 2(1-c) &= 1 - d. \end{aligned} \tag{4}$$

Equations (2) and (4) are three equations with three unknowns. Solving these equations gives us the VNM solution.

With a small adjustment, these equations can give us the equilibrium solution for different bet sizes. They also let us solve the problem of different hand distributions for the two players, for example if Player 1's hand is dealt from $U(0.05, 0.9)$, and Player 2's hand is dealt from $U(0, 0.85)$. With more elaboration, they can be used to solve a two-person game where both players have the option of betting, as in normal poker [**2**].

In this paper we will use the simplicity of indifference equations to allow us to examine betting strategies for other forms of poker.

4. ADDING VOLATILITY. Our main topic in this paper is the effect of hand volatility on betting strategy. In the VNM game, each player already has their final hand—it's not going to change. But in real poker, most of the betting takes place in the early rounds, when there are still more cards to be dealt.

There are many ways that hand values can change. More cards can be added to the hand (stud poker); you can exchange some cards for others (draw poker); community cards can be dealt (Texas Hold'em, which we will call Holdem, as well as Omaha); or cards can suddenly become wild (Follow the Queen). The net effect is that relative hand value changes over the course of the game. An elegant paper by Bernasconi et al. [**1**] addresses volatility by adding a step in which the hand values are reversed some of the time. We'll discuss the Bernasconi paper further in Section 8.

We model these changes by enriching the VNM model to make a game we call VNMplus. In VNMplus, before any showdown, there is an additional move of Nature. Each player gets a random number between 0 and r. This is added to their original

$U(0, 1)$ number. The player with the lowest total wins the pot. Neither player knows their final hand during the betting round. If we set $r = 0$, we get the original VNM game. There is more volatility when r is larger, so we call r the "relative volatility" number.

Definition 4.1. *VNMplus is the VNM game, with the addition of a random $U(0, r)$ deal before the showdown. We call r the "relative volatility."*

To illustrate how this can affect bluffing, we start with a case of very large r. The extreme case of $r = 100$ is similar to $r = \infty$, but can more easily be dealt with numerically.

We switch terminology now, referring to Player X and Player Y. It's not important for this example, but later we'll use Player X for the player whose exact hand x is known, and Player Y for the player whose hand is known to occupy a range $[0, y]$.

For this example, Player X begins with known hand x and at showdown time has a hand represented by the probability density function $X(t)$ corresponding to the uniform distribution $U(x, x + 100)$. Player Y begins with known hand y and at showdown time has a hand represented by the probability density function $Y(s)$ for the uniform distribution $U(y, y + 100)$. These functions for the case $x < y$ are illustrated in Figure 5.

The probability of Player X getting a hand better than or equal to a particular hand h is $\int_{-\infty}^{h} X(t)\, dt$. The probability of Player Y getting a hand better than Player X is given by the double integral

$$\int_{-\infty}^{\infty} X(t) \int_{-\infty}^{t} Y(s)\, ds\, dt. \tag{5}$$

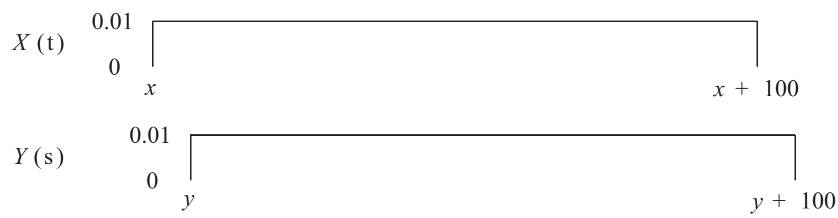

Figure 5. X(t) and Y(s) for the very high volatility game

For the $Y(s)$ we're looking at, assuming $t \le y + 100$, the inner integral is $\int_{-\infty}^{t} Y(s)\, ds = \int_{y}^{t} 0.01\, ds = 0.01(t - y)$. The outer integral, when $x < y$, becomes

$$\int_{y}^{x+100} (0.01)(0.01)(t - y)\, dt$$

which is

$$\frac{1}{2} + 0.01(x - y) + \frac{1}{2}(0.01)(0.01)(x - y)^2.$$

The worst case for Player Y is when $y = 1$ and $x = 0$, which gives Player Y winning chances of 0.49005. Therefore, each player, X or Y, has a probability of winning a showdown that is slightly more or slightly less than $\frac{1}{2}$, depending on who has the better starting hand.

Theorem 4.2. *For the equilibrium solution of VNMplus with* $r = 100$*, there is no bluffing.*

Proof. For Player 2's strategy, we observe that calling is always better than dropping,

$$u_2(h_2,\ \text{call}) = -1 + 3P[h_2 \text{ wins showdown}] \geq 0.47015,$$
$$u_2(h_2,\ \text{drop}) = 0.$$

For Player 1's strategy, we see that betting is better than calling when his chance of winning the showdown is greater than 50%, i.e., when his hand is less than 0.5,

$$u_1(h_1,\ \text{check}) = 1P[h_1 \text{ wins showdown}],$$
$$u_1(h_1,\ \text{bet}) = -1 + 3P[h_1 \text{ wins showdown}].$$

The strategy when $r = 100$ is

$$s_1 = (1)$$
$$s_2 = (0.5, 1). \qquad \blacksquare$$

In other words, Player 1 has no motivation to bluff with a bad hand, because there is no chance that Player 2 will drop.

5. REALISTIC VOLATILITY. With no volatility, $r = 0$, bluffing is a key part of Player 1's strategy. With extremely high volatility, $r = 100$, bluffing should be avoided altogether. There must be some point, $0 < r < 100$, where the strategy of bluffing disappears. We call this special value R the volatility threshold. This value is useful in practical terms. For a game situation with volatility greater than R, we know not to bluff with bad hands. For situations with volatility less than R, bluffing adds to our return.

Definition 5.1. *R threshold. For VNMplus, there must be some value between* $r = 0$ *and* $r = 100$ *where bluffing disappears from Player 1's strategy. We call this value* R.

We can use indifference equations to calculate R. We introduce the notation $p_1(h_1, h_2)$ to represent the probability that Player 1 wins a showdown when he has starting hand h_1 and Player 2 has starting hand h_2. When Player 1 bets with hand b, and Player 2 calls, we know that Player 2 has a hand in the range $[0, c]$. We use the notation $p_1(b, [0, c])$ to represent the probability of Player 1 winning, conditional on $h_1 = b$ and $h_2 \in [0, c]$. With this notation, equations (1) become

$$-1 + 2(1-c) + 3p_1(b, [0, c]) = p_1(b, [0, 1]),$$
$$-1 + 2(1-c) + 3p_1(d, [0, c]) = p_1(d, [0, 1]),$$
$$-1 + \frac{3}{b+1-d}(b(1 - p_1([0, b], c)) + (1-d)(1 - p_1([d, 1], c))) = 0. \qquad (6)$$

We'll come back to these equations, or their equivalent using the p_y notation introduced below, throughout the paper. In Sections 5 and 6 we find the volatility threshold R. In Section 7 we move on to other values of r, and in Sections 8 through 11 we discuss various applications to real life poker games.

For our first task, finding the threshold R, we're looking for the point where d, Player 1's marginal bluff, equals exactly 1. So d is a constant. The variables are b, c, and r. The indifference equations are

$$\begin{aligned} -1 + 2(1-c) + 3p_1(b, [0, c]) &= p_1(b, [0, 1]), \\ -1 + 2(1-c) + 3p_1(1, [0, c]) &= p_1(1, [0, 1]), \\ -1 + 3(1 - p_1([0, b], c)) &= 0. \end{aligned} \tag{7}$$

Note that r does not appear in these equations. It arises from the calculation of p_1. At this point we need to bring back our Player X, Player Y terminology. In each of the p_1 functions, one of the players has a fixed hand x, while the other player has a hand in the range $[0, y]$. In the first two equations, Player 1 is Player X. In the third equation, he is Player Y.

This leads us to introduce a new notation, $p_y(x, [0, y], r)$. This function represents the probability that Player Y will win a VNMplus showdown when Player X has hand x, Player Y has a hand in the range $[0, y]$, and the relative volatility is r. Using p_y, the indifference equations are

$$\begin{aligned} -1 + 2(1-c) + 3(1 - p_y(b, [0, c], r)) &= 1 - p_y(b, [0, 1], r), \\ -1 + 2(1-c) + 3(1 - p_y(1, [0, c], r)) &= 1 - p_y(1, [0, 1], r), \\ -1 + 3(1 - p_y(c, [0, b], r)) &= 0. \end{aligned} \tag{8}$$

To calculate p_y, we begin with the double integral (5) we used for the high volatility case. Restricting the integrals to their nonzero ranges gives us the integral

$$p_y(x, [0, y], r) = \int_x^{x+r} X(t) \int_0^t Y(s)\, ds\, dt. \tag{9}$$

As in the high volatility case, $X(t)$ is the probability distribution of $U(x, x+r)$. $Y(s)$ is more interesting. It is the probability distribution of a convolution of $U(0, y)$ and $U(0, r)$. These distribution functions are illustrated in Figures 6 and 7.

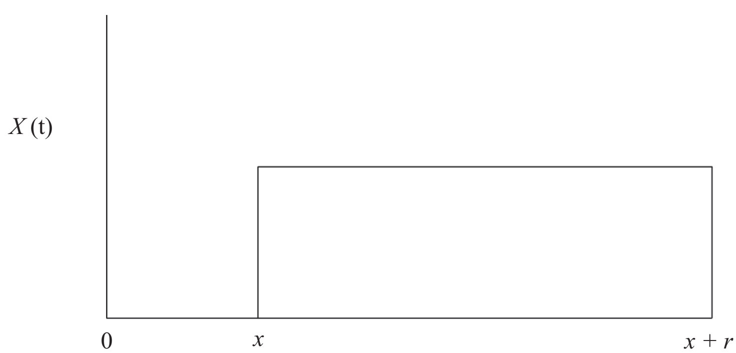

Figure 6. X(t) probability density function

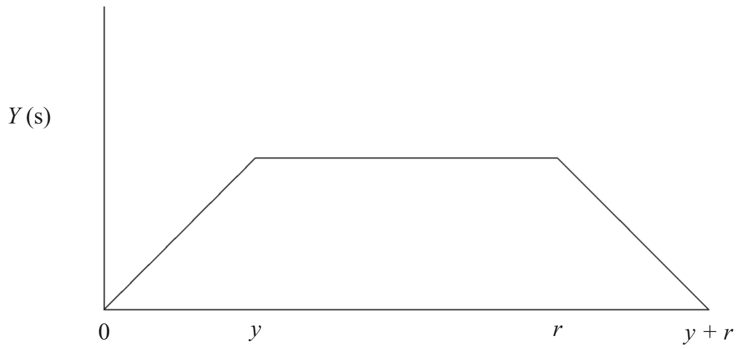

Figure 7. Y(s) probability density function

Unfortunately, the evaluation of this double integral is dependent on the relative values of x, y, and r. Different situations give rise to fourteen different integrations. We enumerate these cases and show how to evaluate two of them in the Appendix. The two cases we need for discovering the R threshold are $y < x < r$, which we refer to as p_{ylo}, and $x < y < r$, which we refer to as p_{yhi}. Once again restating the indifference equations, we get

$$\begin{aligned} -1 + 2(1-c) + 3(1 - p_{ylo}(b, [0, c], r)) &= 1 - p_{yhi}(b, [0, 1], r), \\ -1 + 2(1-c) + 3(1 - p_{ylo}(1, [0, c], r)) &= 1 - p_{ylo}(1, [0, 1], r), \\ -1 + 3(1 - p_{yhi}(c, [0, b], r)) &= 0. \end{aligned} \tag{10}$$

The evaluation of p_{ylo} and p_{yhi} in the Appendix gives

$$\begin{aligned} p_{ylo}(x, [0, y], r) &= \frac{1}{2r^2}(r^2 - x^2 + xy + 2xr - yr) - \frac{y^2}{6r^2}, \\ p_{yhi}(x, [0, y], r) &= \frac{(2x + r - y)}{2r} + \frac{(y-x)^3 - x^3}{6yr^2}. \end{aligned} \tag{11}$$

A simple sanity check for these equations is to look at the limit as r approaches infinity, when each player should have an equal chance of winning a showdown. In this limit, we indeed find $p_{ylo}(x, [0, y], r) = p_{yhi}(x, [0, y], r) = \frac{1}{2}$.

Another sanity check is when $x = y$. For that case, $p_{ylo}(x, [0, y], r) = p_{yhi}(x, [0, y], r) = (3r^2 + 3xr - x^2)/6r^2$. To verify that this makes sense, we can set $x = y = 0$. Both players start with a fixed hand of 0, and so both should have an equal chance of winning. The equation evaluates to $\frac{1}{2}$.

6. SOLVING THE EQUATIONS. Plugging (11) into (10) gives us three equations with three unknowns, b, c, and r. Unfortunately the equations are too cumbersome for an analytical solution, as far as we can see. The alternative is to find a numerical solution. Spreadsheets such as Excel and Open Office Calc (and packages like Matlab) have algorithmic solvers that can be used to solve such problems. It is important to use a nonlinear solver for these nonlinear equations, and to put appropriate limits on the variables. The Excel spreadsheet that we used to solve for the volatility threshold R is available online as Supplemental Material.

By entering equations (11) and (10) into the nonlinear solver, and setting the limits $0 \le c \le b \le 1$ and $b \le r$ we get the solution

$$
\begin{aligned}
b &= 0.462, \\
c &= 0.440, \\
r &= 1.084,
\end{aligned} \tag{12}
$$

which is unique.

Claim 6.1. *The R threshold, at which bluffing disappears (for a bet size equal to the pot), is at $r = 1.084$. You should have bluffing as part of your strategy when r is less, and never bluff when r is greater than this value.*

This is the solution we've been looking for. But it's not obvious how to make use of it. We'll discuss in Section 8 what this means for real life games such as Holdem.

7. OPTIMAL/EQUILIBRIUM STRATEGIES FOR OTHER VALUES OF R. We have calculated the R threshold. In so doing, we developed techniques that can be used to calculate equilibrium strategies for any value of r. We can solve any game of VNMplus by making r a constant and using our spreadsheet solver to determine b, c, and d.

For example, if you have a practical poker case where $r = 0.71$, you can solve VNMplus to find the strategy $s_1 = (0.268, 0.933)$, and $s_2 = (0.431)$. Rather than following this exact strategy, you would use it to diagnose your opponent's weaknesses, and take advantage. If your opponent was following this equilibrium strategy, they would be calling about 43% of the time. If they call less than that, you can take advantage by bluffing more than you otherwise would. If, on the other hand, you think your opponent is bluffing more than 6.7% of the time (100% − 93.3%), you should call more than the equilibrium amount.[1]

For calculations such as this, where $r \neq R$, you need various p_y integrations , shown in Table A.1 of the Appendix. The procedure is this: choose a value of r, and use a solver to combine the appropriate p_y expression with the indifference equations (6). Let the solver calculate b, c, and d. This is how we calculated the $r = 0.71$ case, as well as all the plots and other results in the remainder of this paper. For the plots, we also use a computer program (available in Supplemental Material) to calculate Player 1's expected return for any given set of strategies.

Experimenting with a variety of cases shows that for most values of r, the equilibrium solution has $b < c$. But for values near the R threshold, $b > c$. It would be interesting to know why.

Since we do not have a general proof of 3.1, it would be nice to have the VNMplus equivalent of Figure 4 to reassure ourselves that our indifference equation solutions are in fact Nash equilibria.

A reasonable example is the case $r = 0.71$, discussed above. (Any value of r where bluffing is useful would serve equally well.) Figure 8 shows Player 1's expected return for both of his actions (bet or check) for each starting hand, when Player 2 is using this strategy. The graph makes it apparent that he does best by betting with hands better than 0.268 and bluffing with hands worse than 0.933. Conversely, Figure 9 shows that Player 2 does best by calling with hands better than 0.431 and dropping (for a return of zero) with worse hands, when she is confronted by Player 1's strategy.

[1] There is a difference here between VNM and VNMplus. For a zero volatility game, when your opponent is bluffing too much, it is profitable to call with any hand that can beat a bluff. For VNMplus, this is not the case. If, for example, $s_1 = (0.268, 0.90)$, it would not be profitable to call with $h_2 = 0.89$.

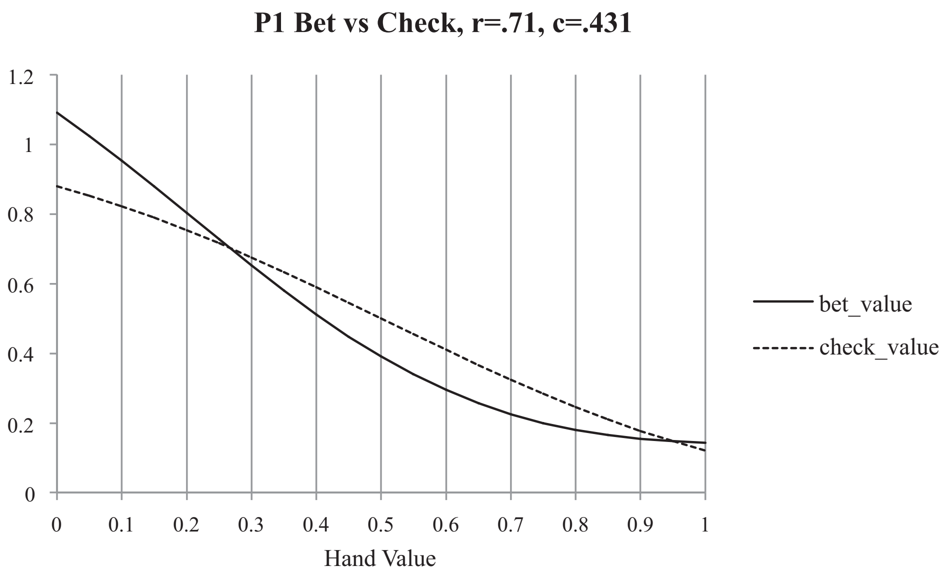

Figure 8. P1's return, betting vs checking with various hands

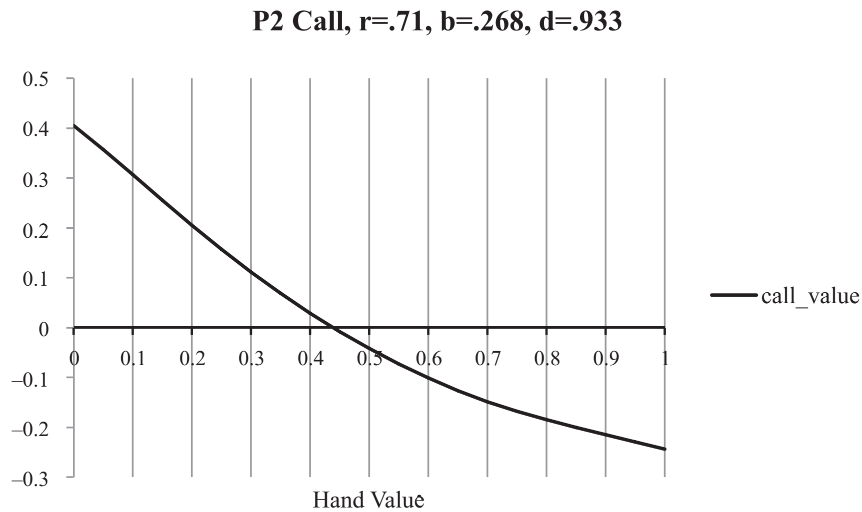

Figure 9. P2's return, always calling

8. VOLATILITY AND HOLDEM. It would be useful to find a way to apply this to real games, like Holdem. We need a way to relate the volatility of community cards to our abstract VNMplus volatility. Also, unlike VNMplus, Holdem has four rounds of betting, and the bet size is usually not equal to the pot size.

Holdem terminology and play:

- Two face-down cards dealt to each player (players look at their own cards);
- First round of betting;
- Three face up cards dealt to middle of table, "the flop";
- Second round of betting;

- Another face up card;
- Third round of betting;
- Final face up card;
- Fourth round of betting;
- Players make best hand from combination of two face-down cards and five face-up (community) cards. Best hand wins.

In a subgame perfect Nash equilibrium for Holdem, there would be a continuation value of the game for each player at the game node just before the flop is dealt. That is to say that the game has a theoretical value for each player depending on the contents of their hands and on the amount of money in the pot. The equilibrium strategy for the first round of betting should depend on these continuation values, and on the amount of volatility in the rest of the game.

To study real life Holdem, we use a simplified version that we call FlopShow_N.

Definition 8.1. *FlopShow_N $(2 \leq N \leq 6)$ is defined as follows:*

- *2 face down cards dealt to each player;*
- *$N - 2$ face up cards dealt to the middle;*
- *One round of betting;*
- *$7 - N$ face up cards dealt to the middle, "the flop";*
- *Players make best hand from combination of two face down cards and five face up cards. Best hand wins.*

FlopShow_N is played with a real 52 card deck. It uses the normal hand valuations for poker (e.g., three-of-a-kind is better than two pair), which we will not enumerate.

FlopShow_2 is an approximation of the first round of betting in Holdem. Each player has two face down cards with five face up cards still to be dealt. FlopShow_5 is an approximation of the second round of betting in Holdem (two face down, three face up, two to be dealt). FlopShow_6 is an approximation of the third round. (FlopShow_3 and FlopShow_4 would approximate nonexistent states of Holdem, with one or two face up community cards.) If we can figure out a betting strategy for FlopShow_N, it will be a good guide to betting in Holdem.

The measurement of volatility in VNMplus is simply the value r. For FlopShow_N and Holdem we need a quantitative measure for the amount of change that can be introduced by cards still to be dealt. A general measure of volatility that is useful for all three games comes from asking the question, "How often do relative hand values remain unchanged by the deal?" We call this metric HOM (Hand Order Maintained). For a small r value, or with few cards to be dealt, hand values usually hold up, so HOM is a little less than 1. For large r values, or with many cards to be dealt, hand values change a lot, so HOM is slightly greater than 0.5.

We claim, without proof, that if VNMplus and FlopShow_N have the same HOM, they have the same equilibrium strategy.

Definition 8.2. *HOM (Hand Order Maintained) is the probablity that the best hand before cards are dealt is still the best hand after cards are dealt.*

We calculate HOM for VNMplus in a straightforward way. For any value of r, simulate many rounds of VNMplus, and count the number of times that hand order is maintained. We do this with a short computer program, available as Supplemental

Material. Having run this for many values, we noticed that the relationship between r and HOM looks somewhat exponential. We did a curve fitting exercise for the region of most interest, $0 \le r \le 4$, and found $HOM = 0.5628 + 0.4372e^{-0.8314r}$. While this approximation is numerically close, we doubt any analytical accuracy.

Calculation of HOM for FlopShow$_N$ is conceptually simple. The most straightforward technique is to deal out many hands from a physical deck, and count the results. We began by doing this, but got tired after several hundred hands. To get large amounts of data, one must resort to computer simulation. Any simulation requires you to deal with the question of which hand is best before the flop. In this case, "best" means most likely to win after the flop. You can find tables for this online. They tell you, for example, that in a two player game, an Ace-King of the same suit is better than a pair of sevens, but worse than a pair of eights.

At this point, having assumed a relationship between VNMplus and FlopShow$_N$, and a relationship between FlopShow$_N$ and Holdem, we can ask questions about Holdem strategy. The question that got us started on this inquiry was, "Is it profitable to bluff in the first round of betting in Holdem?" This translates to the question, "Does the equilibrium strategy for FlopShow$_2$ include bluffing?"

Our physical deck dealing and computer simulations of FlopShow$_2$ revealed that $HOM \approx 61\%$. That corresponds to VNMplus $r = 2.85$, which is far greater than the volatility threshold $R = 1.084$ where bluffing becomes unprofitable.

Claim 8.3. *In Holdem it is unprofitable to bluff in the first round of betting.*

If you have an opponent who bluffs in the first round of Holdem, you should call more often than you otherwise would.

We leave other Holdem questions for the reader to explore. But we'll mention that our FlopShow$_5$ simulation yielded an r value close to the R threshold, suggesting that bluffing is of limited use in the second round of betting. Our FlopShow$_6$ simulation gave $r \approx 0.65$, which says that bluffing is necessary in the third round of betting in Holdem. For the final round of betting, see our earlier paper [**2**].

9. BERNASCONI'S COIN FLIP. Bernasconi et al. [**1**] took an approach that is similar to ours. Like us, they add volatility to the original VNM game. They do this by introducing a biased coin flip that occurs after the betting and before the showdown. With some probability $0 \le q \le \frac{1}{2}$, the values of the hands are reversed. A higher q value corresponds to more volatility in our scheme. In Section 8 we introduced the "HOM" (hand order maintained) metric. It is always the case that $\frac{1}{2} \le HOM \le 1$. The HOM of Bernasconi's game equals $1 - q$.

Among other things, Bernasconi explores the interesting fact that the value of the game to Player 1 rises as q rises between zero and $\frac{1}{3}$. At $q = \frac{1}{3}$ there is a maximum, then the game value declines as q increases to $\frac{1}{2}$. In short, a small amount of uncertainty, in their scheme, makes the game better for Player 1.

Surprisingly, under our scheme, the opposite is true. For small amounts of relative volatility, that is to say when HOM is close to but less than one, the game value to Player 1 is less. A look at the difference between the r of VNMplus and the q of Bernasconi's game explains this. The basic difference is that Bernasconi's coin flip q will reverse the value of two hands regardless of their original value. But relative volatility r is more likely to reverse the hands if their original values are similar. For widely different hand values, a small r has no effect at all.

For example, in the game with no volatility, as Von Neumann showed, the equilibrium strategy is $s_1 = (\frac{2}{9}, \frac{8}{9})$, $s_2 = (\frac{4}{9})$. For a small q value, say 0.1, the final hand

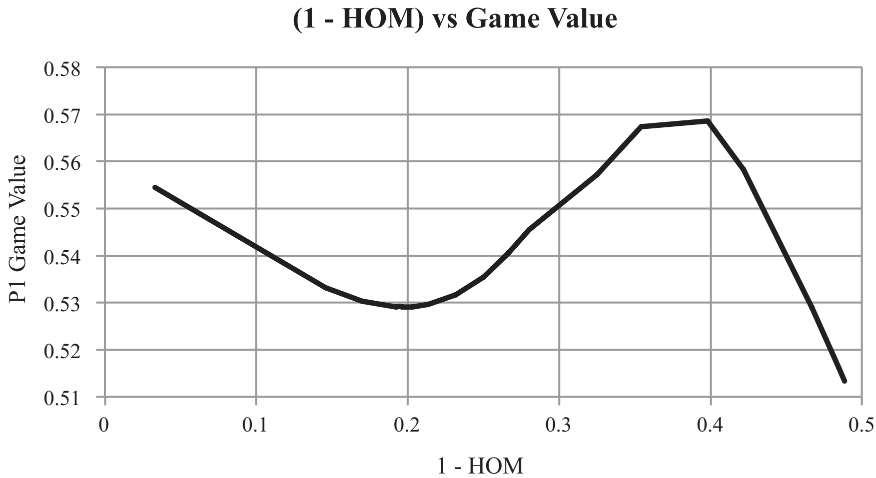

Figure 10. VNMplus game value for increasing volatility, as measured by $1 - HOM$

values are swapped 10% of the time. This is an advantage to Player 1, because his bluffing hands, which otherwise have no chance of winning, now become winners 10% of the time. Using our scheme, if we set $HOM = 0.9$, it corresponds to an r value of 0.33. This has no effect on Player 1's bluff hands. They still lose every time, because $d - c$ is greater than 0.33. However, this r value increases the return for Player 2's calling hands that are greater than b. With no volatility, these hands would always lose to legitimate bets $h_1 < \frac{2}{9}$. With $r = 0.33$, they win a fair percentage of the time.

These differences between the coin flip q method and the relative volatility r method means that relative volatility more closely models standard poker games such as Holdem. However, the coin flip method is probably a better model for games that have occasional sudden reversals, such as Follow the Queen.

Bernasconi et al. chart the value of their VNM variation for various values of q. We do the same for VNMplus. Our calculations let us see the value of the game for the equilibrium strategy at various values of r, see Figure 10. When $HOM = 1$ (no volatility) the original game has a value of $\frac{5}{9}$. With $HOM = 0.5$, infinite volatility, the value is 0.5. In between, there is a local minimum at $HOM \approx 0.8$ ($r = 0.74$) and a maximum at $HOM \approx 0.62$ ($r = 2.5$).

10. DIFFERENT BET SIZES. We'd like to review our definition of bluffing. It means to bet with your worst hands, in the hope that the opponent who has a hand better than yours will drop, and that the opponent's drops will bring you more money than not betting with these hands. (Compare this with the semibluff, discussed below.) In Holdem terms, this might mean betting or raising with a 7 and 2 of different suits (a very bad hand) in the first round. Your hope would be that either the opponents will drop immediately, or they will drop as you continue to bluff in future rounds, or that the board will fall lucky for you. A large part of your expected value must come from winning immediately in the first round.

What Claim 8.3 says is that hand values in the first round are so likely to change, that your opponents probably have a profitable call. But those calculations were made assuming a pot-size bluff. One might say, "Sure, they won't be scared off by a pot-size bluff. But I'm playing no limit. I can represent a better hand with a huge bet, and that will scare them off." Or one might surmise that smaller bet sizes will make a bluff profitable.

The same spreadsheet solver we used before can be used to factor in the effect of different size bets. The initial pot size is still 1 but the bet size becomes β. The new equations are

$$-\beta + (1+\beta)(1-c) + (1+2\beta)(1 - p_{ylo}(b, [0,c], r)) = 1 - p_{yhi}(b, [0,1], r),$$
$$-\beta + (1+\beta)(1-c) + (1+2\beta)(1 - p_{ylo}(1, [0,c], r)) = 1 - p_{ylo}(1, [0,c], r),$$
$$-\beta + (1+2\beta)(1 - p_{yhi}(c, [0,b], r)) = 0. \tag{13}$$

Putting these equations through the solver gives the interesting result that the no-bluff threshold value R actually decreases for large bets. When $\beta = 5$, representing a bet five times the size of the pot, R goes down to 0.91.

One case of large β corresponds exactly to a real-world situation that comes up often in head-to-head (i.e., two player) tournament play. This situation is analyzed by Chen and Ankenman, in their excellent book *The Mathematics of Poker* [**3**]. They discuss a specific case of tournament poker[2] for which the best strategy for both players is what they call "jam or fold." This means you should either bet all your money (jam) or drop out of the hand (fold). The situation they analyze (two player tournament play, where one player is close to elimination), is different from our situation (two players with infinite money). Even so, their equilibrium solution says that Player 1 should bet 58% of the time, with no bluffing. Our equilibrium solution says that with their bet size, and a volatility of 2.85 (our calculated r for FlopShow$_2$), Player 1 should bet 55% of the time, with no bluffing.

Significantly smaller bets also lower the R threshold. For $\beta = 0.3$, $R = 1.03$. So for large or small bets, it takes less volatility to make bluffing unprofitable. The highest value of R is not when $\beta = 1$ however. The highest threshold value is when $\beta = 0.784$ and $R = 1.087$. With that bet size, the strategy is nicely symmetrical, $s_1 = (0.5, 1)$, $s_2 = (0.5)$. It would be interesting to know why this is.

11. POKER CONCEPTS. The uselessness of bluffing in the first round of Holdem (and arguably in the second round as well) tells us a lot about how the game should be played. A successful player has to have a paradoxical mindset: in the early rounds *never* bet with bad cards; in the later rounds *frequently* bet with bad cards. This is the meaning of the phrase "tight-aggressive," which is often used by advice books to teach the proper approach to the game.

Fortunately, removing bluffing from your early round strategy doesn't mean giving up deception. Two other techniques, slow play and semibluff, become more important when true bluffing is ineffective. By its definition, slow play is a mirror of true bluffing. A slow play is to underbet your very best hands. A true bluff is to overbet your very worst hands. The idea of slow play is to penalize aggression by the other players, letting them raise the pot with inferior hands. It can be used effectively at any point in the hand, early or late.

But in strategic terms the opposite of a true bluff is a semibluff. A semibluff (which we first encountered in Sklansky [**5**]), is betting with a hand that is fairly good, but probably not the best hand in the pot. If your "normal" strategy would be to bet with all hands better than b, the semibluff has you betting with the best of your normally nonbetting hands. (Contrast this to a true bluff, which has you betting with the worst of your normally nonbetting hands.) A semibluff is helpful when a hand has good potential, such as a strong straight or flush draw, especially when facing a probable

[2]One player has a stack that's equal to only 10 times the big blind, or counting the small blind, 6.67 times the total ante.

pair. It can pay off in various ways: win the pot outright, build the pot for when you make your hand, buy a free card for another chance to make your hand, or set the stage for a true bluff in a later round. The semibluff is most useful in the early rounds of betting, when there is still a lot of volatility. This is precisely when a true bluff is useless. The true bluff is most useful in the late rounds of betting, when there is little or no volatility. This is precisely when a semibluff is useless.

The technique of using VNMplus to calculate a volatility threshold can be used with other poker games. Sometimes these analyses can have surprising results, which will be the subject of other papers than this one. But sometimes they just reconfirm something that experienced players are already aware of. For example, when playing Holdem with more than two players, you need a better hand to bet or call as the number of players goes up. Unsurprisingly, VNMplus simulations show that the volatility threshold is lower when there are more players. This is a corollary to the idea that when there are more players to be forced out of the pot, a smaller percentage of bluffs are successful.

12. CONCLUSIONS. VNM provided early insight into poker strategy by establishing that the bluff is a mathematical technique. We discuss a simplified way to reproduce their calculations, which enables us to add some variations. We changed the game by adding a random number draw after the betting round.

When we increase the size of the random number to a certain point, which we call the relative volatility threshold R, the technique of bluffing becomes unprofitable. We show how to calculate R for a modified VNM game.

By using a simplified variation of Holdem, we are able to come to some conclusions about the real-world game. In particular, it's unprofitable to bluff in the first betting round of Holdem.

Appendix. The p_y integral has a different solution depending on the relative values of x, y, and r. There are fourteen total cases. We give details of the integration for the cases $x < y < r$, which we call p_{yhi}, and $y < x < r$, which we call p_{ylo}.

For p_{yhi}, the outer integral can be broken into three integrals

$$\int_x^{x+r} = \int_x^y + \int_y^r + \int_r^{x+r} .$$

For each one, we'll calculate the area under the $Y(t)$ curve using geometry.

For t between x and y the graph of $Y(s)$ is the triangle shown in Figure 11 with a width of t and a height of $\frac{1}{r}\frac{t}{y}$. Its area is $\frac{t^2}{2yr}$.

For t between y and r the area under the $Y(t)$ graph is the area shown in Figure 12. It is a triangle with width y and height $1/r$ plus the rectangle with width $t - y$ and height $1/r$. The result is $\frac{2t-y}{2r}$.

For t greater than r the area is one minus that of the triangle at the right of Figure 13. That has width $r + y - t$ and height $(\frac{1}{r})\frac{r+y-t}{y}$ which gives us $1 - \frac{(r+y-t)^2}{2yr}$.

For all values of t between x and $x + r$, $X(t) = \frac{1}{r}$ so our first integral $\int_x^y \frac{1}{r}\frac{t^2}{2yr}dt$ evaluates to $\frac{1}{6yr^2}(y^3 - x^3)$.

The second integral $\int_y^r \frac{1}{r}\frac{1}{2r}(2t - y)dt$ evaluates to $\frac{1}{2r}(r - y)$.

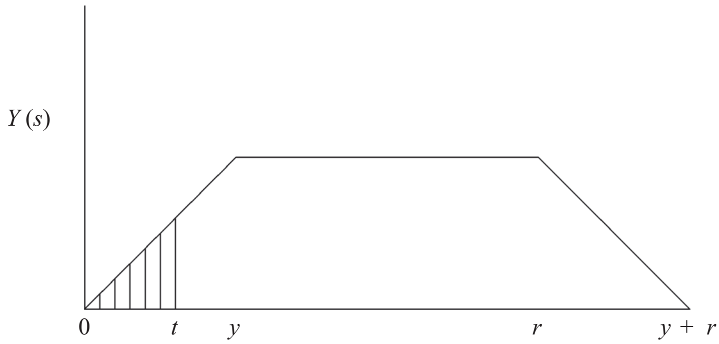

Figure 11. Integral of Y(s) when $t < y$

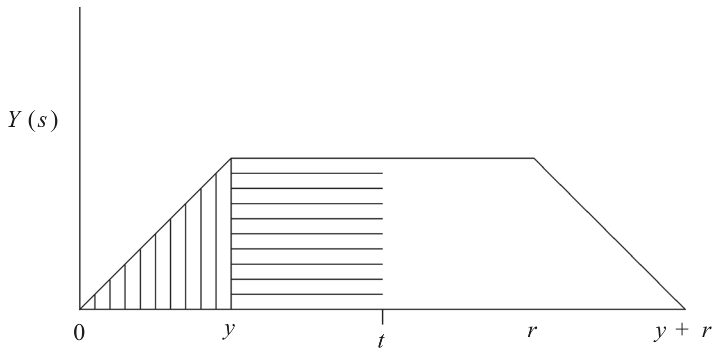

Figure 12. Integral of Y(s) when t is between y and r

The third integral $\int_r^{r+x} \frac{1}{r}(1 - \frac{(r+y-t)^2}{2yr})dt$ can be broken into two pieces

$$\int_r^{r+x} \frac{1}{r} dt - \int_r^{r+x} \frac{(r+y-t)^2}{2yr^2} dt.$$

The second part benefits from a change of variable $u = r + y - t$. We end up with $\frac{x}{r} - \frac{1}{6yr^2}(y^3 - (y-x)^3)$.

Adding the three integrals, we get

$$p_{yhi}(x, [0, y], r) = \frac{(2x + r - y)}{2r} + \frac{(y-x)^3 - x^3}{6yr^2}. \tag{14}$$

Similarly, for p_{ylo}, we break the integral into three parts:

$$\int_x^{x+r} = \int_x^r + \int_r^{y+r} + \int_{y+r}^{x+r}.$$

Since $x > y$, the first $Y(s)$ integral here is the same as the second integral above, that is, $\frac{(2t-y)}{2r}$. Then,

$$\int_x^r \frac{1}{r}\frac{1}{2r}(2y - t)dt = \frac{1}{2r^2}(r^2 - x^2 + yx - yr).$$

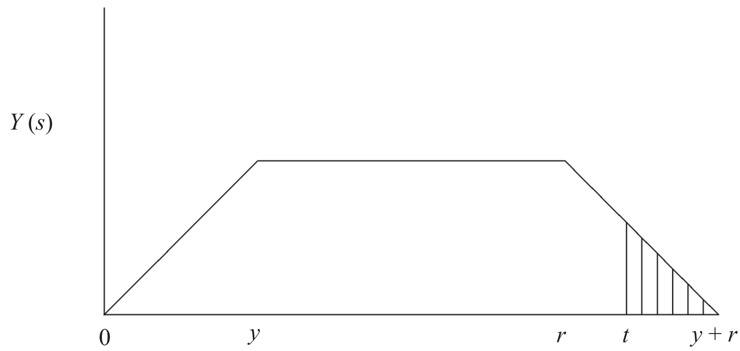

Figure 13. When t is greater than r, subtract the triangle from 1

Table A.1. p_y integrals for various cases.

Case	p_y
$x < y < r$	$\frac{(2x+r-y)}{2r} + \frac{(y-x)^3 - x^3}{6yr^2}$
$x < r < y < x+r$	$\frac{2x+r-y}{2r} + \frac{(y-x)^3 - x^3}{6yr^2}$
$x < r < x+r < y$	$\frac{r^3 - x^3 + 3rx^2 + 3r^2x}{6yr^2}$
$r < x < y < x+r$	$\frac{x+r-y}{r} + \frac{(x+y-r)(y-x)}{2yr} + \frac{(y-x)^3 - r^3}{6yr^2}$
$r < x < x+r < y$	$\frac{x}{y}$
$y < x < r$	$\frac{1}{2r^2}(r^2 - x^2 + xy + 2xr - yr) - \frac{y^2}{6r^2}$
$y < r < x < y+r$	$1 - \frac{(y+r-x)^3}{6yr^2}$
$y < r < y+r < x$	1
$r < y < x < y+r$	$1 - \frac{(y+r-x)^3}{6yr^2}$
$r < y < y+r < x$	left as exercise
$x < y_0 < y < r$	$\frac{2y^3 + y_0^3 - 3y^2y_0}{12r^2(y-y_0)} + \frac{(x+r-y)(x+r-y_0)}{2r^2}$
$x < r < y_0 < y < x+r$	$\frac{y^3 - y_0^3 + 3(x+r)(y-y_0)(x+r-y-y_0)}{6r^2(y-y_0)}$
$x < r < y_0 < x+r < y$	$\frac{(x+r)^3 - y_0^3 + 3y_0^2(x+r) - 3y_0(x+r)^2}{6r^2(y-y_0)}$
$x+r < y_0$	0

The second $Y(s)$ integral for the $y < x$ case matches the third integral in the $x < y$ case, $1 - \frac{(r+y-t)^2}{2yr}$. Then,

$$\int_r^{y+r} \frac{1}{r}\left(1 - \frac{(r+y-t)^2}{2yr}\right)dt \text{ (using the same change of variables)} = \frac{y}{r} - \frac{y^2}{6r^2}.$$

For the third integral, when $x > y+r$, the $Y(s)$ integral is the total area under the curve, i.e., 1. Then,

$$\int_{y+r}^{x+r} \frac{1}{r} dt = \frac{1}{r}(x-y).$$

Adding the three integrals gives

$$p_{ylo}(x, [0, y], r) = \frac{1}{2r^2}(r^2 - x^2 + xy + 2xr - yr) - \frac{y^2}{6r^2}. \tag{15}$$

Integrals for the other cases are evaluated similarly. There are five simple cases for which $x < y$: one in which r is greater than both x and y, two in which r is between them, and two in which r is less than either. There are five similar cases for which $y < x$. One of these is a low volatility case that is left as an exercise for the reader.

There are four cases for which the bottom of the y range is not zero. These cases come up when there is bluffing, and we want to evaluate $p_y(c, [d, 1], r))$ in the third line of equations (6). In these cases, the bottom of the y range, y_0, appears in the case specification and in the p_y result. Please go to www.maa.org/amm_supplements to find an Excel spreadsheet that has examples and directions for the calculation of equilibrium strategies in games with volatility, and small utility programs written in Ruby, to calculate some results for the paper.

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