Lecture 6

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Ideas used in Lecture 5.

- Transition matrix $\mathbf{P} = (p_{ij})$ on a state space **States** $= \{i, j, k, \ldots\}$.
- Definition of Markov chain $(X_t, t = 0, 1, 2, \ldots)$.
- Simple examples of Markov chains.

We now start to develop the math theory of Markov chains. This material is in Chapter 3 of textbook [PK].

Write $\mu_i(t) = \mathbb{P}(X_t = i)$ and regard $\mu(t) = (\mu_i(t), i \in \textbf{States})$ as a row-vector. The relationship between $\mu(t)$ and $\mu(t-1)$ is

$$\mu_j(t) = \sum_i \mu_i(t-1)p_{ij}.$$

In vector-matrix notation this is

$$\mu(t) = \mu(t-1)\mathbf{P}.$$

So inductively on t

$$\mu(t) = \mu(0)\mathbf{P}^{(t)}, \quad \mathbf{P}^{(t)} = \mathbf{PP}\ldots\mathbf{P} \text{ (t times matrix multiplication).}$$

This implies that the individual entries $p_{ij}^{(t)}$ of $\mathbf{P}^{(t)}$ have the meaning

$$p_{ij}^{(t)} = \mathbb{P}(X_t = j | X_0 = i).$$

$$p_{ij}^{(t)} = \mathbb{P}(X_t = j | X_0 = i) \quad t\text{-step transition probabilities.}$$

$$\mu(t) = \mu(0)\mathbf{P}^{(t)}.$$

Question: So what is the interpretation of column-vectors? What does

$$(*) \quad h(t) = \mathbf{P}^{(t)} h$$

mean for a column vector $h = (h_j, j \in \textbf{States})$?

Answer: Think of h as a function $h : \textbf{States} \to \mathbb{R}$ and then consider

$$h_i(t) = \mathbb{E}(h(X_t) | X_0 = i).$$

Then the vector $h(t) = (h_i(t), i \in \textbf{States})$ is given by (*).

Calculating $\mathbf{P}^{(t)}$ with pencil-and-paper is only possible in very simple or special cases. Consider a 2-state chain

$$p_{01} = a,\ p_{00} = 1 - a; \quad p_{10} = b,\ p_{11} = 1 - b$$

with $0 < a, b < 1$. Here we can calculate [on board]

$$\mathbb{P}(X_t = 0 | X_0 = 0) = \tfrac{b}{a+b} + \tfrac{a}{a+b}(1 - a - b)^t.$$

For **theory**, we are given a Markov chain $(X_0, X_1, \ldots)$ on **States** with a given transition matrix $\mathbf{P}$.

Theory: Conditioning on first step.

The method we used for simple symmetric random walk works for a Markov chain. For a subset $A \subset$ **States** consider the **hitting time**

$$T_A = \min\{t \geq 0 : \ X_t \in A\}$$

Maybe never hit, in which case we define $T_A = \infty$; for now assume we know $T_A < \infty$. We study

$$h(i) = \mathbb{E}(T_A | X_0 = i) = \mathbb{E}_i T_A$$

introducing notation $\mathbb{E}_i(\cdot)$ and $\mathbb{P}_i(\cdot)$ for initial state i. By conditioning on the first step,

$$(i \in A) : \ h(i) = 0$$

$$(i \notin A) : \ h(i) = 1 + \sum_j p_{ij} h(j). \tag{1}$$

Consider disjoint subsets A, B of **States** – maybe $A = \{a\}$ and $B = \{b\}$. Let's study

$$g(i) = \mathbb{P}_i(T_A < T_B)$$

the probability starting at i of hitting A before hitting B. We have

$$g(i) = 1, \ i \in A; \quad g(i) = 0, \ i \in B \tag{2}$$

and by conditioning on the first step

$$g(i) = \sum_j p_{ij} g(j), \ i \notin A \cup B. \tag{3}$$

Do these equations have a unique solution? Logically, the chain either hits A before B, or hits B before A, or never hits either. So consider

$$g^*(i) = \mathbb{P}_i(\text{ never hit } A \text{ or } B)$$

for which the corresponding equations are

$$g^*(i) = \sum_j p_{ij} g^*(j), \ i \notin A \cup B; \quad g^*(i) = 0, \ i \in A \cup B.$$

If g solves equations (2,3) then so does $g + g^*$ and so the solution is not unique, unless $g^* \equiv 0$, that is unless the chain **always** hits A or B.

Conceptual point: often we answer problems by writing down equations and then solving them, but to be more precise we should check the solution is unique.

In the particular case of previous slide, one can prove that, provided

$$\mathbb{P}_i(\text{ never hit } A \text{ or } B) = 0$$

there is indeed a unique solution.

I will do one numerical example – Exercise 3.4.3 of [PK] – on the board.

Here is another simple example.

Throw a fair die until getting two 6's in succession. This requires some random number T of throws. Calculate $\mathbb{E}T$.

This can be done by defining a Markov chain as follows.
$X_t = 0$ if the t'th throw is not 6.
$X_t = 1$ if the t'th throw is 6 but the previous throw was not 6.
$X_t = 2$ if the t'th throw is 6 and the $(t-1)$ throw was 6.
This chain has states $\{0, 1, 2\}$ and

$$p_{00} = 5/6, \ p_{01} = 1/6, \ p_{10} = 5/6, \ p_{12} = 1/6.$$

We can set up and solve [on board] the equations for

$$h(i) = \mathbb{E}_i T_2.$$

We find $h(1) = 36$ and $h(0) = 42$, and the answer is the 42.