Lecture 38

David Aldous

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Standard Brownian motion $(B(t), 0 \le t < \infty)$ has the properties

- Sample paths $t \to B(t)$ are continuous
- $B(t)$ is a martingale
- $B^2(t) - t$ is a martingale.

Levy's theorem states that Brownian motion is the **only** process with these properties. In other words, to check that a given process is Brownian motion, we don't need to check the "Normal distributions and independent increments" properties in the definition; instead we can just check the properties above, for a filtration $(\mathcal{F}(t), 0 \le t < \infty)$.

We will work more intuitively in terms of "infinitesimal increments". For a process $X(t)$ write the increment $X(t + dt) - X(t)$ as $dX(t)$. The two martingale properties above can be rewritten as

- $\mathbb{E}(dB(t)|\mathcal{F}(t)) = 0$
- $\mathbb{E}((dB(t))^2 \,|\mathcal{F}(t)) = dt$.

Suppose we are given two functions $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$. It can be shown that (under minor assumptions on these functions) there exists a **unique** process $(X(t), 0 \le t < \infty)$ with the properties

- Sample paths $t \to X(t)$ are continuous
- $\mathbb{E}(dX(t)|\mathcal{F}(t)) = \mu(X_t)dt$
- $\mathbb{E}((dX(t))^2 \,|\mathcal{F}(t)) = \sigma^2(X_t)dt$

In other words the process $(X(t), 0 \leq t < \infty)$ is the continuous-time Markov process specified by

- Sample paths $t \to X(t)$ are continuous
- $\mathbb{E}(dX(t)|X(t) = x) = \mu(x)dt$
- $\mathrm{var}\ (dX(t)\ |X(t) = x) = \sigma^2(x)dt$

This is analogous to specifying a Markov chain by specifying its transition matrix. Such processes are called **diffusions**.

Standard Brownian motion $B(t)$ is the case $\mu(x) \equiv 0,\ \sigma(x) \equiv 1$. Recall that for constants μ, σ we can define $X(t) = \mu t + \sigma B(t)$; this is the case $\mu(x) \equiv \mu,\ \sigma(x) \equiv \sigma$. In this case $X(t)$ has a Normal distribution but in general it does not.

Textbooks develop the theory of such processes – there are formulas for hitting probabilities and mean hitting times, for instance. In this lecture I will just show how diffusions can arise as scaling limits of discrete Markov processes, analogous to the way we introduced Brownian motion as the scaling limit of simple random walk.

Example: Wright-Fisher model with mutation

- k genes per generation
- each gene is allele A or allele B
- each gene is a copy of a uniform random gene from the previous generation, except that ...
- A mutates to B with probability α/k
- B mutates to A with probability β/k

Write $X_n^{(k)}$ for number of A alleles in generation n.

We want to rescale the process by considering the **proportion** of genes that are A, and to take 1 generation as a time interval δ in "rescaled time units" – δ depends on k and we will calculate later what it is. So the rescaled process is

$$Y_{n\delta}^{(k)} = k^{-1} X_n^{(k)}.$$

To find the diffusion which is the rescaled limit (as $k \to \infty$), what we need to do is to calculate (to first order) the change in mean and variance in one step (generation, in this example) of the discrete process, then rescale.
In this example we have [board]

$$\mathbb{E}(X_1^{(k)} - x | X_0^{(k)} = x) = 0 - \frac{\alpha}{k} x + \frac{\beta}{k}(k - x)$$

$$\mathrm{var}\ (X_1^{(k)} - x | X_0^{(k)} = x) \approx k \frac{x}{k} \frac{k-x}{k} + 0.$$

Restating this in terms of $Y = X/k$ we see [board]

$$\mathbb{E}(Y_\delta^{(k)} - y | Y_0^{(k)} = y) \approx \delta(-\alpha y + \beta(1 - y))$$

$$\mathrm{var}\ (Y_\delta^{(k)} - y | Y_0^{(k)} = y) \approx \delta y(1 - y)$$

where we have chosen $\delta = 1/k$. This says that the process $(Y_t^{(k)})$ is approximately the diffusion with

$$\mu(y) = -\alpha y + \beta(1 - y), \quad \sigma^2(y) = y(1 - y)$$

whose state space is $[0, 1]$.

Ehrenfest urn model.

- $2k$ balls, two boxes.
- Pick uniform random ball, more to other box.

Study $X_n^{(k)}$ = number of balls in left box after n steps. We know the stationary distribution is Binomial$(2k, 1/2)$.

Rescale by defining

$$Y_{n\delta}^{(k)} = (X_n^{(k)} - k)/\sqrt{k}$$

for δ to be calculated later.

To find the diffusion which is the rescaled limit, what we need to do is to calculate (to first order) the change in mean and variance in one step of the discrete process, then rescale.

$$\mathbb{E}(X_1^{(k)} - x | X_0^{(k)} = x) = \tfrac{k-x}{k}$$

$$\mathrm{var}\ (X_1^{(k)} - x | X_0^{(k)} = x) \approx 1 \text{ for } x = k \pm O(k^{1/2}).$$

and then rescaling give [board]

$$\mathbb{E}(Y_\delta^{(k)} - y | Y_0^{(k)} = y) \approx -\delta y$$

$$\mathrm{var}\ (Y_\delta^{(k)} - y | Y_0^{(k)} = y) \approx \delta$$

for $\delta = 1/k$.

This says that the process $(Y_t^{(k)})$ is approximately the diffusion with

$$\mu(y) = -y, \quad \sigma^2(y) = 1$$

whose state space is $(-\infty, \infty)$. This is the **Ornstein-Uhlenbeck process**.

Here is the first interesting piece of "theory" for diffusions. Let (Y_t) be the diffusion with given drift and variance rate functions $\mu(y)$, $\sigma^2(y)$. Let f be a smooth strictly increasing function $\mathbb{R} \to \mathbb{R}$. Then $X_t = f(Y_t)$ is also a diffusion, and we can calculate its functions

$$\widehat{\mu}(x) = f'(y)\mu(y) + \tfrac{1}{2} f''(y)\sigma^2(y)$$
$$\widehat{\sigma}(x) = f'(y)\sigma(y)$$

where $x = f(y), \ y = f^{-1}(x)$.

This allows us to use martingale arguments to calculate hitting probabilities. [board]