Lecture 36

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The M/G/1 queue model is

- Customers arrive at times of a rate-λ Poisson point process
- Service times $Y_1, Y_2, \ldots$ are IID.
- Write $\nu = \mathbb{E}Y$ and note that $1/\nu$ is "service rate".
- $X(t) =$ number of customers at time t.
- 1 server.

Here $(X(t), 0 \leq t < \infty)$ is **not** a continuous-time Markov chain. But we can do some calculations.

Here is a third method which calculates something different. There is a **discrete-time** Markov chain associated with the M/G/1 queue:
$X_n =$ number of customers just after the departure of the n'th customer.

Here [board]

$$X_{n+1} = \max(X_n - 1, 0) + A_n$$

$$A_n = \text{ number of arrivals during next service period}$$

and the A_n are IID.

As a special property of the M/G/1 queue, the stationary distribution of (X_n) is the same as the equilibrium distribution of "number of customers" in the original queue process. Using this fact we can calculate the expectation of "number of customers".

[calculation on board – follows [PK] sec. 9.3.1].

The M/G/∞ queue model is

- Customers arrive at times of a rate-λ Poisson point process
- Service times $Y_1, Y_2, \ldots$ are IID.
- Service starts immediately (infinite number of available servers)
- $X(t) =$ number of customers at time t.

Easy to analyze as follows. Customer i arrives at some time T_i and has some service time Y_i; we can represent (T_i, Y_i) as a Poisson process in $\mathbb{R}^2$ with rate

$$\lambda(t, y) = \lambda f(y); \quad f(y) \text{ is density of } Y.$$

[board]
Starting empty, distribution of $X(t)$ is Poisson with mean $\lambda \int_0^t \mathbb{P}(Y \geq s) ds$. So in $t \to \infty$ limit the mean is $\lambda \mathbb{E} Y$.

Example: airport parking lot. [not in text]

Cars arrive at (large) rate λ and remain for Exponential(1) times. The numbers of parked cars form a $M/M/\infty$ queue and the stationary distribution is Poisson(λ). But there is also a "spatial" aspect; imagine parking spaces numbered $1, 2, 3, \ldots$ and each arriving car parks in the lowest-numbered empty space.
Question: when "you" arrive, you park in some space U: what is the distribution of U?

Answer. Fix $0 < u < 1$ and consider

$$N_u = \text{ number of empty spaces among spaces } [1, u\lambda].$$

This "number of empty spaces" process is approximately a $M/M/1$ queue with "arrival" rate $u\lambda$ and "service" rate λ. So at stationarity

$$\mathbb{P}(N_u = 0) = 1 - u, \quad \mathbb{E}N_u = \tfrac{u}{1-u}.$$

$$\mathbb{P}(U \le u\lambda) = \mathbb{P}(N_u \ge 1) \approx u, \ 0 < u < 1.$$