Lecture 29

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6 November 2015

We have seen two kinds of "purely random" process:

- coin-tossing, dice-throwing, etc – IID sequences.
- Poisson process of random points in d dimensions.

A third kind one might imagine is

- a "purely random" continuous function.

It is not obvious what this means. To start, consider simple symmetric random walk

$$S_n = \sum_{i=1}^{n} \xi_i; \quad \mathbb{P}(\xi_i = \pm 1) = 1/2.$$

For large n, how can we draw the graph of $(S_0 = 0, S_1, \ldots, S_n)$ on "unit size" paper?

[board]

Know $\mathbb{E}S_n = 0$ and s.d.$(S_n) = \sqrt{n}$.

The picture suggests that the "scaling limit" [jargon!] of the random walk is a process $(B(t), 0 \leq t < \infty)$ with the property

1. $B(t)$ has Normal(0,t) distribution.

It turns out there is a mathematical object called "standard Brownian motion" (BM) with properties (1) and

2. $B(t) - B(s)$ has Normal(0,t-s) distribution $(s < t)$.

3. For $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k$ the increments $B(t_1) - B(s_1), \ldots, B(t_k) - B(s_k)$ are independent.

4. The sample paths $t \to B(t)$ are (random) continuous functions of t.

Keep in mind this is a **model** – looking at some time-varying real-world quantity, it may or may not behave like this BM model.

One example which does fit the model quite well is short-term stock prices [show].

As the stock price graphs suggest, although continuous the sample paths are irregular – not differentiable.

BM is important because

1. one can do many explicit calculations.
2. it is a "building block" for defining other random processes.

This part of the course is more "mathematical" in the sense of emphasizing calculations rather than toy models.

Here is a very simple instance of (1).

Given parameters $-\infty < \mu < \infty$ and $0 < \sigma < \infty$ we can define

$$X(t) = \mu t + \sigma B(t)$$

called *Brownian motion with drift rate μ and variance rate σ^2*. In particular $X(1)$ has Normal(μ, σ^2) distribution.

The appearance of the Normal distribution is not arbitrary (cf. the Poisson distribution in the Poisson process), as shown by the next theorem.

Theorem

If a process $X(t), 0 \le t < \infty$ has $X(0) = 0$, continuous sample paths, and (for each $a > 0$) the increments $X(a), X(2a) - X(a), X(3a) - X(2a), \ldots$ are IID, then the process must be BM with some drift and variance rates.

This is a consequence of the CLT.

Explicit formulas for BM will ultimately be based on the Normal density formula, but first let us see some "structural properties" of BM. Write $=_d$ for "equal in distribution".

1: Symmetry. $(B(t), 0 \le t < \infty) =_d (-B(t), 0 \le t < \infty)$.

2: Markov. Given t_0 and the "past" $(B(t), 0 \le t \le t_0)$, the "future" process $\widetilde{B}(u) = (B(t_0 + u) - B(t_0), 0 \le u < \infty)$ has the distribution of BM and is independent of the past process.

3: Scaling. For $c > 0$ the "scaled" process
$\widetilde{B}(u) = (c^{-1/2} B(cu),\ 0 \le u < \infty)$ has the distribution of BM.
[show Wikipedia demo]

4: Martingale. $(B(t), 0 \le t < \infty)$ is a martingale.

Recall facts about the Normal density.

$$\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2), \ -\infty < z < \infty$$

is the density of a "standard Normal(0, 1)" RV Z. Write the distribution function as

$$\mathbb{P}(Z \leq z) = \Phi(z) = \int_{-\infty}^{z} \phi(x) dx.$$

We have

$$\mathbb{E}Z = 0, \quad \mathrm{var}\ (Z) = 1.$$

A few lines of calculus show [board]

$$\mathbb{E}|Z| = \sqrt{2/\pi}.$$

Calculations with $B(t)$ easily done by scaling: $B(t) =_d t^{1/2} Z$ and so (for instance) $\mathbb{E}|B(t)| = \sqrt{2t/\pi}$. The general relation

$$f_{cY}(y) = c^{-1} f_Y(y/c)$$

gives the density of $B(t)$

$$f_{B(t)}(x) = t^{-1/2} \phi(t^{-1/2} x) = (2\pi t)^{-1/2} \exp(-x^2/(2t)), \ -\infty < x < \infty.$$