Lecture 27

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Theorem (Optional Sampling Theorem)

If (X_t) is a martingale and τ is a stopping time, then (under extra technical conditions)

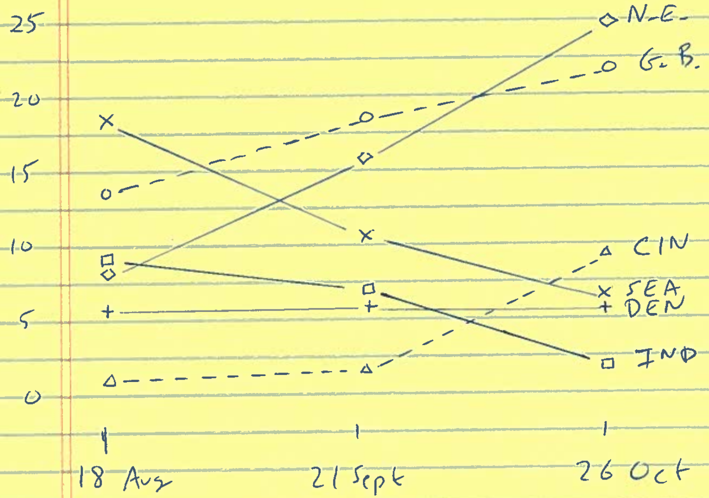
$$\mathbb{E}X_\tau = \mathbb{E}X_0.$$

Advanced Probability courses give different versions of the "extra technical conditions" – see [BZ] Theorem 3.1 for one version of these conditions. In the examples I will give, it is not hard to show the conditions hold.

The "double when you lose" strategy shows that some extra condition is necessary. [board]

Conceptual point: The Optional Sampling Theorem and the previous "gambling systems" theorem constitute an informal "conservation of fairness" principle: the overall results of any "system" based on fair games is like a single fair bet. Even in models not explicitly involving gambling, one can do calculations by inventing hypothetical gambling strategies and using this principle.

Outcome	PredictWise	Derived Betfair Price	Betfair Back	Betfair Lay
New England Patriots	25 %	$ 0.241	4.10	4.20
Green Bay Packers	22 %	$ 0.211	4.70	4.80
Cincinnati Bengals	9 %	$ 0.093	10.50	11.00
Seattle Seahawks	7 %	$ 0.073	13.00	14.50
Carolina Panthers	6 %	$ 0.061	16.00	17.00
Denver Broncos	6 %	$ 0.060	16.50	17.00
Arizona Cardinals	5 %	$ 0.053	18.50	19.50
Atlanta Falcons	4 %	$ 0.038	26.00	27.00
Pittsburgh Steelers	3 %	$ 0.031	30.00	34.00
Indianapolis Colts	2 %	$ 0.021	46.00	50.00
New York Giants	2 %	$ 0.021	44.00	50.00
Philadelphia Eagles	2 %	$ 0.020	48.00	55.00
New York Jets	2 %	$ 0.019	50.00	55.00
Minnesota Vikings	1 %	$ 0.015	65.00	70.00
Dallas Cowboys	1 %	$ 0.014	65.00	80.00
St Louis Rams	1 %	$ 0.011	75.00	110.00
Miami Dolphins	1 %	$ 0.008	110.00	150.00
Washington Redskins	0 %	$ 0.006	100.00	700.00
Oakland Raiders	0 %	$ 0.005	160.00	350.00
New Orleans Saints	0 %	$ 0.004	200.00	350.00
San Diego Chargers	0 %	$ 0.004	150.00	550.00
Baltimore Ravens	0 %	$ 0.002	400.00	1,000.00
Kansas City Chiefs	0 %	$ 0.002	350.00	1,000.00
Buffalo Bills	0 %	$ 0.002	300.00	1,000.00
Chicago Bears	0 %	$ 0.002	500.00	1,000.00


25
20
15
10
5
0
N.E.
G.B.
CIN
SEA
DEN
IND
18 Aug
21 Sept
26 Oct

[from Lecture 1]

Just for fun, here are probabilities (as perceived by gamblers) for next Superbowl winner.

No simple math theory for the actual numbers here (can build complex statistical models based on players statistics for past performance) but there is math theory for how these probabilities will fluctuate as the season progresses:

there will be (on average) 5 teams whose perceived probability will sometime exceed 20%.

I'll explain this in the "martingales" section of the course.

there will be (on average) 5 teams whose perceived probability will sometime exceed 20%.

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Initially each team's probability was less than 20% . So consider the strategy:

Place a 20 unit bet (fair odds) on each team, at the moment (if ever) its probability reaches 20%.

[continue on board]

(Here we assume probabilities change as a continuous function, not quite realistic).

[show relevant slides from popular talk]

[Link to paper on course web page]

[show Predictit] [show Ladbrokes] [show "real clear" poll]