Lecture 11

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The circle of 4 results from previous Lectures.

Define the **return time**

$$T_i^+ = \min\{t \geq 1 : X_t = i\}.$$

Fix a reference state b.

Theorem

Suppose irreducible.
(a) If state space is finite then $\mathbb{E}_b T_b^+ < \infty$.
(b) Suppose $\mathbb{E}_b T_b^+ < \infty$. *Define*

$$a(b,i) = \mathbb{E}_b \sum_{s=1}^{T_b^+} \mathbb{1}_{(X(s)=i)}$$

$=$ *mean number of visits to* i *before returning to* b. *So* $a(b,b) = 1$. *Then*

$$\pi_i = \frac{a(b,i)}{\mathbb{E}_b T_b^+}$$

is a stationary distribution, and is the **only** *stationary distribution.*

Theorem

If the chain is irreducible, positive-recurrent and aperiodic, then for any initial distribution

$$\mathbb{P}(X(t) = j) \to \pi_j \text{ as } t \to \infty$$

where π is the unique stationary distribution.

Theorem

Write

$$N_i(t) = \sum_{s=0}^{t-1} \mathbb{1}_{(X(s)=i)} = \text{ number of visits to } i \text{ before } t.$$

If the chain is irreducible and positive-recurrent, then for any initial distribution

$$t^{-1} N_i(t) \to \pi_i \text{ a.s. } \text{ as } t \to \infty.$$

Note this implies but is stronger than previous fact

$$\mathbb{E}[t^{-1} N_i(t)] \to \pi_i \text{ as } t \to \infty.$$

A final result is rather subtle. Note we only need this when the state space is infinite.

Proposition

If irreducible, if there exists a probability distribution π *satisfying* $\pi = \pi \mathbf{P}$, *then the chain is positive-recurrent.*

Then we can apply previous theorems and this π is the unique stationary distribution.

See texts for proofs. I want to focus on what these results say, in our specific examples.

An important setting where we can do explicit calculations is **birth-and-death chains** (note [PK] calls these "general random walk".) Here the state space is either $\{0, 1, \ldots, N\}$ or $\{0, 1, 2, \ldots\}$ and the transitions are of the form

$$p_{i,i+1} = p_i > 0; \quad p_{i.i-1} = q_i > 0; \quad p_{ii} = 1 - p_i - q_i \geq 0$$

when i is not an endpoint of the state space. There are two different cases for endpoints.
In the **absorbing** case we set $p_{00} = 1$, and (finite case) $p_{NN} = 1$.
In the **reflecting** case we set $p_{01} = p_0 > 0, \ p_{00} = 1 - p_0$, and (finite case) $p_{N,N-1} = q_N > 0, \ p_{N,N} = 1 - q_N$.

We first consider the reflecting case. Here the chain is irreducible. So by our Theorems, in the finite case the chain has a stationary distribution. We can calculate the stationary distribution by solving the detailed balance equations.
[board]

Conclusion. Define $w_0 = 1$ and

$$w_i = \prod_{j=0}^{i-1} \frac{p_j}{q_{j+1}}, \qquad w = \sum_{i \geq 0} w_i.$$

If $w < \infty$, which is certain in the finite case, then the chain is positive-recurrent and the stationary distribution is

$$\pi_i = w_i / w, \ \ i \geq 0.$$

Note we used the final Proposition in the infinite case.

If $w = \infty$ one can show [outline on board] the chain is not positive-recurrent.

A simple special case is where $p_i = p, q_i = 1 - p$ for all i. Here $w_i = (p/1 - p)^i$. So for $p < 1/2$, we see π is the shifted Geometric(θ) distribution for $\theta = 1 - \frac{p}{1-p}$.

Now consider the absorbing case. This is [PK] Problem 3.6.1 with slightly different setup.

[board]

Conclusion. for $0 \le i \le N$,

$$\mathbb{P}_i(T_N < T_0) = \frac{D(i)}{D(N)}, \quad D(i) = \sum_{j=1}^{i} \prod_{k=1}^{j-1} \frac{q_k}{p_k}.$$

One can also calculate, in a similar way, a formula for the mean time $\mathbb{E}_i \min(T_0, T_N)$ – see [KP].

The stationary chain

Consider an irreducible positive-recurrent chain – we know it has a unique stationary distribution π. When we start the chain with initial distribution π we call it the **stationary chain** $(X_t, t = 0, 1, 2, , \ldots)$ and the starting point for all this theory was that, for the stationary chain,

$$X_t \text{ has distribution } \pi \text{ for each } t \geq 0.$$

In fact something stronger is true; for each fixed k,

$$(X_t, X_{t+1}, \ldots, X_{t+k}) \text{ has the same distribution for each } t \geq 0.$$

There is a mental picture "watching a movie" of the process; if we start watching at time t, the stationary property is that the statistical properties of what we see do not depend on the starting time t.

There are several interesting features of the stationary chain. Consider

$$U = \min\{t \geq 1 : X_t = X_0\}.$$

Then [board]

$$\mathbb{E}U = \text{ (number of states)}.$$

Next, let's imaging running the movie backwards. That is, fix N, consider the stationary chain $(X_t, 0 \le t \le N)$ and then define

$$(X_0^*, X_1^*, \ldots, X_N^*) = (X_N, X_{N-1}, \ldots, X_0).$$

So X_0^* has distribution π and we calculate [board]

$$\mathbb{P}(X_1^* = j | X_0^* = i) = \pi_j p_{ji} / \pi_i.$$

By extending this argument we can show that $(X_t^*, 0 \le t \le N)$ is itself a stationary chain with transition matrix $\mathbf{P}^*$, for

$$p_{ij}^* = \pi_j p_{ji} / \pi_i.$$

In general $\mathbf{P}^*$ is different from $\mathbf{P}$ but it might be the same. We see

$$\mathbf{P}^* = \mathbf{P} \text{ if and only if } \pi_i p_{ij} = \pi_j p_{ji} \ \forall i, j$$

and this was the *detailed balance* condition. Chains with this property are often called **reversible**.