Lecture 2

Very simple problems can be done in the "bare hands": way

- write down all possible outcomes
- $P(A)$ = probability event A happens = sum of the probabilities of the outcomes that make A happen.

For more complicated problems we need to use "rules of probability". Where do these rules come from?

There are two different math connections between Probability and Proportion.

1. Uniform random sampling from a finite set.

2. Law of Averages in repeatable chance experiment.

Take as an axiom that the "rules of probability" are the same as the "rules of proportion".

PROPORTIONS

Set $S = \{\text{students in this room}\}$.
Typical subsets

$$A = \{\text{men}\}; \; B = \{\text{Stat majors}\}; \; C = \{\text{CS majors}\}.$$

For any subset A write $PR(A) = \#A/\#S$.
Numerical values of $PR(A), \; PR(B), \; PR(C) \ldots$ are empirical facts. But logic/arithmetic tells us some general rules.

1. $0 \leq PR(A) \leq 1$, for any A.
2. $PR(A) + PR(A^c) = 1$, any A; here $A^c =$ complement.
3. $PR(B \cup C) = PR(B) + PR(C)$ if B, C <u>disjoint</u>.
4. For any A, B

$$PR(A \cup B) = PR(A) + PR(B) - PR(A \cap B).$$

A partition of S is a list of subsets, such that each element of S is in exactly one of the subsets. So for $S = \{\text{students in this room}\}$, one partition is

$$\{\text{women}\}, \quad \{\text{men}\}$$

and another partition (assuming no double majors) would be

$$\{\text{undeclared}\}, \ \{\text{Stat majors}\}, \ \{\text{CS majors}\}, \ldots$$

Conditional proportion $PR(A|B)$ means "proportion of B who are also A". For instance $PR(\text{women}|\text{Stat majors})$ is the proportion of Stat majors who are women. As a formula

$$PR(A|B) = \#(A \cap B)/\#B = PR(A \cap B)/PR(B).$$

Rule of average conditional proportions

$$PR(A) = PR(A|B_1)PR(B_1) + \ldots + PR(A|B_n)PR(B_n)$$

for any subset A and any **partition** of subsets $B_1, \ldots, B_n$.

Starting point of mathematical probability is

Axiom: the probabilities of different events (within a given chance experiment) satisfy the same general rules as do the proportions of different subsets (within a given set).

The analog of conditional proportion is conditional probability:

$P(A|B)$ means "probability that A happens, if we know that B happens".

The formula is

$$P(A|B) = P(A \text{ and } B)/P(B).$$

It is important to know that $P(A|B)$ and $P(B|A)$ are different; easiest to remember an example using proportions:

- the proportion of men who are doctors
- the proportion of doctors who are men

are very different numbers.
[board: connection between $P(A|B)$ and $P(B|A)$]

There are two different connections between probability and proportion.

1: simple random sampling. A special type of chance experiment is to pick one element of a set "at random", meaning that each element is equally likely to be picked. In this chance experiment

$$P(\text{random element in A}) = PR(A)$$

for any subset A.

[Examples: random students; roulette]

2. Law of averages in repeatable experiments.
Imagine an irregular die; chance of landing "6" may not be 1/6.
Common-sense-law-of-averages says:
if you throw the die N times, calculate the (random) number
$PR_N(6) =$ proportion of the N throws which land "6",
then there should be a limit

$$\lim_{N\to\infty} PR_N(6) = P(6)$$

and we can regard the limit number $P(6)$ as "probability this die lands 6".
In general, for any repeatable chance experiment, and events $A, B, C \ldots$, one can imagine repeating N times and calculating the empirical proportions

$$PR_N(A),\ PR_N(B), PR_N(C), \ldots$$

and we expect these to get closer and closer to the probabilities

$$P(A),\ P(B), P(C), \ldots$$

Because the numbers $PR_N(\cdot)$ satisfy the rules of proportion, the limit probabilities $P(\cdot)$ must also.

Example using proportions

A University has only 2 Grad Depts

	Dept E	Dept F
applicants	100	400
accepted	20	200

What proportion of all applicants are accepted?
Easy by counting: $\frac{220}{500} = 44\%$.

Let's do it using the Rule of average conditional proportions.

[on board]

Example using probabilities

You and I each throw a fair die. What is the chance that your number is within 1 of my number?

Can do by counting: $\frac{16}{36} = 44.4\%$.

Let's do it using the Rule of average conditional probabilities.

[on board]

Example. Deal 2 cards from deck

A	first card is Ace
B	exactly 1 of the 2 cards is an Ace
C	second card is Ace

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= \frac{48}{51} \times \frac{4}{52} + \frac{4}{51} \times \frac{48}{52} \\ &= 14.5\% \end{aligned}$$

$$P(C) = \ ???$$

This asks for the **unconditional** probability of C.

[on board]

Sampling example. Town has 1,000 households containing 1,800 adults. Consider two schemes for picking an adult.
1. List 1,800 names; pick a name at random, equally likely.
2. List 1,000 households. Pick a household at random, equally likely. Then pick an adult from that household, equally likely.

Are these two schemes the same? Suppose John Smith lives alone, and Elizabeth Lee lives with another adult. Then in scheme 2

$$P(\text{pick JS}) = \frac{1}{1000}$$
$$P(\text{pick EL}) = \frac{1}{1000} \times \frac{1}{2} = \frac{1}{2000}.$$

So **not** same as scheme 1.

Example of conditional probability

I have 3 two-sided cards.

X	X
X	blank
blank	blank

I shuffle and show you one side of one card. This turns out to be ____.
What is the chance the other side is X ?

Example
Know a family has 2 children, and know at least one child is a girl. What is the chance the other child is a girl?

There is a long discussion in Wikipedia "Boy or Girl paradox". The brief answer is:

it depends on **how** we got this information.

[say stories in words]

(1) "new baby" story; chance $= 1/2$
(2) Computer printout story; chance $= 1/3$