10/5/2021

Minimax Estimation

Outline

1) Minimax risk, estimator

2) Least favorable priors

Minimax risk

Last idea for choosing an estimator: worst-case risk

$$\underset{\delta}{\text{minimize}} \ \sup_\theta \ R(\theta; \delta)$$

The minimum achievable sup-risk is called the <u>minimax risk</u> of the estimation problem

$$r^* = \inf_\delta \sup_\theta R(\theta; \delta)$$

An estimator δ^* is called <u>minimax</u> if it achieves the minimax risk, i.e.

$$\sup_\theta R(\theta; \delta^*) = r^*$$

Game theory interpretation:

1) Analyst chooses estimator δ
2) Nature chooses parameter θ to max. risk

<u>NB</u>: Nature chooses θ adversarially, not X

Compare to Bayes, where Nature chooses prior from a known distribution

$\Rightarrow$ Nature plays a specific mixed strategy

We will look for Nature's Nash-equil. strategy

Least Favorable Priors

Minimax closely related to Bayes

<u>Key observation:</u> average-case risk $\leq$ worst-case risk

For proper prior Λ, the Bayes risk is

$$r_\Lambda = \inf_\delta \int R(\theta;\delta)\, d\Lambda(\theta)$$
$$\leq \inf_\delta \sup_\theta R(\theta;\delta) = r^*$$

If δ_Λ Bayes then $r_\Lambda = \int R(\theta;\delta_\Lambda)\, d\Lambda(\theta)$

$\Rightarrow$ Bayes risk of <u>any</u> Bayes estimator lower bounds r^*

<u>Least favorable prior</u> Λ^* gives best lower bound: $r_{\Lambda^*} = \sup_\Lambda r_\Lambda$

Sup-risk of any estimator upper bounds r^*

$$\underset{(\text{any } \delta)}{\sup_\theta R(\theta;\delta)} \geq r^* \geq r_{\Lambda^*} \geq \underset{(\text{any } \Lambda)}{r_\Lambda}$$

Can exhibit minimax est. / LF prior by finding δ and Λ that collapse these ineq. to $=$

<u>Theorem</u> If $r_\Lambda = \sup_\theta R(\theta; \delta_\Lambda)$ with Bayes estimator δ_Λ then:

(a) δ_Λ is minimax

(b) If δ_Λ is unique Bayes (up to $\overset{a.s.}{=}$) for Λ, it is unique minimax

(c) Λ is least fav.

<u>Proof</u> a) Any other δ:

$$\sup_\theta R(\theta;\delta) \geq \int R(\theta;\delta)\, d\Lambda(\theta)$$
$$\geq \int R(\theta;\delta_\Lambda)\, d\Lambda(\theta) \qquad (*)$$
$$= r_\Lambda$$
$$= \sup_\theta R(\theta;\delta_\Lambda) \quad \text{by assumption}$$

$\Rightarrow$ r_Λ is minimax risk, δ_Λ is minimax,

b) Replace "$\geq$" with "$>$" in 2^{nd} ineq. $(*)$

c) Any other prior $\tilde{\Lambda}$:

$$r_{\tilde{\Lambda}} = \inf_\delta \int R(\theta;\delta)\, d\tilde{\Lambda}(\theta)$$
$$\leq \int R(\theta;\delta_\Lambda)\, d\tilde{\Lambda}(\theta)$$
$$\leq \sup_\theta R(\theta;\delta_\Lambda) = r_\Lambda \qquad \boxtimes$$

The above theorem gives a checkable condition:

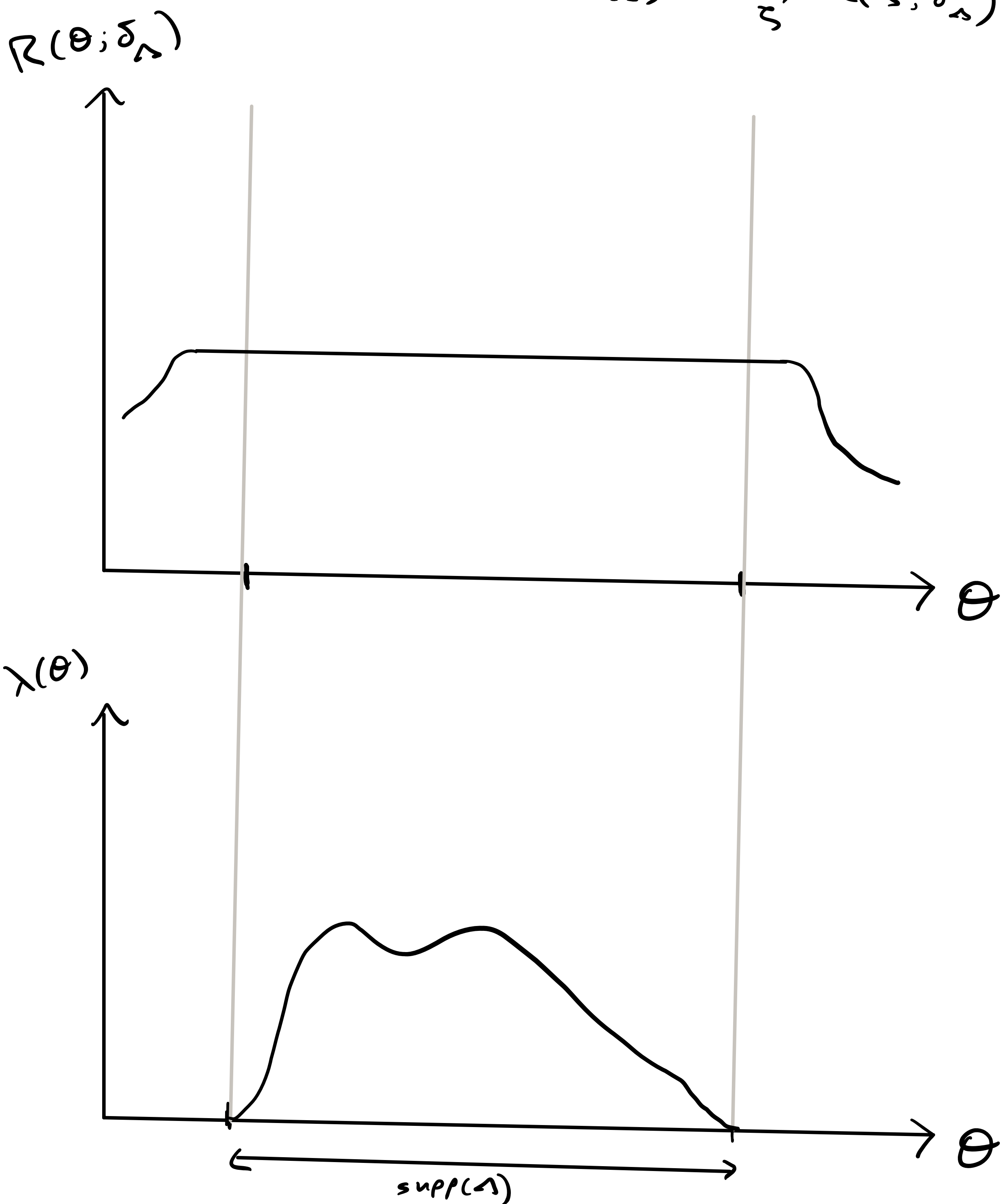
does avg risk = sup risk ?

True if :

mistake on final: saying r_Λ is const. doesn't prove anything

1) $R(\theta; \delta_\Lambda)$ is constant

2) $\Lambda(\{\theta : R(\theta; \delta_\Lambda) = \max_{\xi} R(\xi; \delta_\Lambda)\}) = 1$



Example (Binomial)

$X \sim \text{Binom}(n, \theta)$, estimate θ, sq. err.

Try $\text{Beta}(\alpha, \beta)$, hope to get one with const. risk

$$\delta_{\alpha,\beta}(X) = \frac{\alpha + X}{\alpha+\beta+n}$$

$$R(\theta; \delta_{\alpha,\beta}(X)) = \mathbb{E}_\theta\left[\left(\frac{\alpha+X}{\alpha+\beta+n} - \theta\right)^2\right]$$

$$= \text{Var}_\theta\left(\frac{X}{\alpha+\beta+n}\right) + \left(\frac{\alpha+\theta n}{\alpha+\beta+n} - \theta\right)^2$$

$$= (\alpha+\beta+n)^{-2} \cdot \left[n\theta(1-\theta) + (\alpha - (\alpha+\beta)\theta)^2\right]$$

$$\propto_\theta \underbrace{\left[(\alpha+\beta)^2 - n\right]}_{\text{Set } = 0}\theta^2 + \underbrace{\left[n - 2\alpha(\alpha+\beta)\right]}_{\text{Set } = 0}\theta + \alpha^2$$

Set $(\alpha+\beta)^2 = n$, $2\alpha(\alpha+\beta) = n$

$\Rightarrow \alpha+\beta = \sqrt{n} \Rightarrow 2\alpha\sqrt{n} = n$

$\Rightarrow \alpha = \beta = \sqrt{n}/2$

$\Longrightarrow \text{Beta}\left(\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right)$ is LF

$\dfrac{X + \sqrt{n}/2}{n+\sqrt{n}}$ is minimax ✓ We got lucky!

Question: why so much prior wt. on $\theta = 1/2$?

Least Favorable Sequence

Sometimes there is no least favorable prior, e.g. if par. space isn't compact.

$X \sim N(\theta, 1)$: LF prior should spread mass everywhere, but that is not a proper prior.

Def: A sequence $\Lambda_1, \Lambda_2, \ldots$ is LF if $r_{\Lambda_n} \longrightarrow \sup_{\Lambda} r_{\Lambda}$

Thm: Suppose $\Lambda_1, \Lambda_2, \ldots$ is a prior sequence and δ satisfies $\sup_\theta R(\theta; \delta) = \lim_n r_{\Lambda_n}$

Then a) δ is minimax

b) $\Lambda_1, \Lambda_2, \ldots$ is LF

Proof a) Other est. $\tilde{\delta}$. Then $\forall n$,

$$\sup_\theta R(\theta; \tilde{\delta}) \geq \int R(\theta; \tilde{\delta})\, d\Lambda_n(\theta)$$
$$\geq r_{\Lambda_n}$$

$$\Rightarrow \sup_\theta R(\theta; \tilde{\delta}) \geq \sup_n r_{\Lambda_n}$$
$$\geq \lim_n r_{\Lambda_n}$$
$$= \sup_\theta R(\theta; \delta)$$

b) Prior Λ

$$\begin{aligned} r_\Lambda &= \int R(\theta; \delta_\Lambda)\, d\Lambda(\theta) \\ &\le \int R(\theta; \delta)\, d\Lambda(\theta) \\ &\le \sup_\theta R(\theta; \delta) \\ &= \lim_n r_{\Lambda_n} \end{aligned}$$

$\boxtimes$

<u>Basic Picture:</u>

$\sup_\theta R(\theta; \delta)$ generic δ

$\ge \inf_\delta \sup_\theta R(\theta; \delta)$ $\left(= \sup_\theta R(\theta; \delta^*) \text{ if minimax est. exists}\right)$

$\ge \sup_\Lambda r_\Lambda$ $\left(= r_{\Lambda^*} \text{ if LF prior exists}\right)$

$\ge r_\Lambda$ generic Λ

Bounding minimax risk

Our theorem gives an idea of how to bound r^* for a problem:

<u>Upper bound</u>: If δ is any estimator then

$$r^* \le \sup_\theta R(\theta;\delta) \qquad (= \text{ if } \delta \text{ minimax})$$

<u>Lower bound</u>: If Λ is any prior then

$$r^* \ge \int R(\theta;\delta_\Lambda)\, d\Lambda(\theta) \qquad (= \text{ if } \Lambda \text{ LF})$$

Minimax estimators are very hard to find but minimax bounds are often used in stat theory to characterize hardness (esp. lower)

<u>Ex</u>: Propose practical estimator δ, find Λ for which r_Λ close to $\sup_\theta R(\theta;\delta)$ (or same rate, or cvgs asymptotically)

$\Rightarrow$ Conclude δ can't be improved "much" (*)

<u>Ex</u>: Quantify hardness of a problem by its minimax rate in some asy. régime.

<u>Caveat</u>: A problem might be easy throughout most of par. space but very hard in some bizarre corner you never encounter in practice!