d. Suppose that you hadn't thought of the preceding fact. Explain how you could determine a good approximation to c by generating random numbers on a computer (simulation).

14. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = 2 \times P(H_1)$. As in Section 9.1, the hypothesis H_0 will be chosen if $P(H_0|x) > P(H_1|x)$. For $\sigma^2 = 0.1, 0.5, 1.0, 5.0$:

a. For what values of X will H_0 be chosen?
b. In the long run, what proportion of the time will H_0 be chosen if H_0 is true $\frac{2}{3}$ of the time?

15. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = P(H_1)$. For $\sigma = 1$ and $x \in [0, 3]$, plot and compare (1) the p-value for the test of H_0 and (2) $P(H_0|x)$. Can the p-value be interpreted as the probability that H_0 is true? Choose another value of σ and repeat.

16. In the previous problem, with $\sigma = 1$, what is the probability that the p-value is less than 0.05 if H_0 is true? What is the probability if H_1 is true?

17. Let $X \sim N(0, \sigma^2)$, and consider testing $H_0 : \sigma = \sigma_0$ versus $H_A : \sigma = \sigma_1$, where $\sigma_1 > \sigma_0$. The values σ_0 and σ_1 are fixed.

a. What is the likelihood ratio as a function of x? What values favor H_0? What is the rejection region of a level α test?
b. For a sample, $X_1, X_2, \dots, X_n$ distributed as above, repeat the previous question.
c. Is the test in the previous question uniformly most powerful for testing $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$?

18. Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables from a double exponential distribution with density $f(x) = \frac{1}{2}\lambda \exp(-\lambda|x|)$. Derive a likelihood ratio test of the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda = \lambda_1$, where λ_0 and $\lambda_1 > \lambda_0$ are specified numbers. Is the test uniformly most powerful against the alternative $H_1 : \lambda > \lambda_0$?

19. Under H_0, a random variable has the cumulative distribution function $F_0(x) = x^2$, $0 \le x \le 1$; and under H_1, it has the cumulative distribution function $F_1(x) = x^3$, $0 \le x \le 1$.

a. If the two hypotheses have equal prior probability, for what values of x is the posterior probability of H_0 greater than that of H_1?
b. What is the form of the likelihood ratio test of H_0 versus H_1?
c. What is the rejection region of a level α test?
d. What is the power of the test?

20. Consider two probability density functions on [0, 1]: $f_0(x) = 1$, and $f_1(x) = 2x$. Among all tests of the null hypothesis $H_0 : X \sim f_0(x)$ versus the alternative $X \sim f_1(x)$, with significance level $\alpha = 0.10$, how large can the power possibly be?

21. Suppose that a single observation X is taken from a uniform density on $[0, \theta]$, and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.