To find the value of n that maximizes L_n, consider the ratio of successive terms, which after some algebra is found to be

$$\frac{L_n}{L_{n-1}} = \frac{(n-t)(n-m)}{n(n-t-m+r)}$$

This ratio is greater than 1, i.e., L_n is increasing, if

$$\begin{aligned}(n-t)(n-m) &> n(n-t-m+r)\\ n^2 - nm - nt + mt &> n^2 - nt - nm - nr\\ mt &> nr\\ \frac{mt}{r} &> n\end{aligned}$$

Thus, L_n increases for $n < mt/r$ and decreases for $n > mt/r$; so the value of n that maximizes L_n is the greatest integer not exceeding mt/r.

Applying this result to the data given previously, we see that the maximum likelihood estimate of n is $\frac{mt}{r} = \frac{20 \cdot 10}{4} = 50$. This estimate has some intuitive appeal, as it equates the proportion of tagged animals in the second sample to the proportion in the population:

$$\frac{4}{20} = \frac{10}{n}$$ ■

Proposition B has the following extension.

PROPOSITION C

The number of ways that n objects can be grouped into r classes with n_i in the ith class, $i = 1, \ldots, r$, and $\sum_{i=1}^{r} n_i = n$ is

$$\binom{n}{n_1 n_2 \cdots n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Proof

This can be seen by using Proposition B and the multiplication principle. (Note that Proposition B is the special case for which $r = 2$.) There are $\binom{n}{n_1}$ ways to choose the objects for the first class. Having done that, there are $\binom{n-n_1}{n_2}$ ways of choosing the objects for the second class. Continuing in this manner, there are

$$\frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{(n-n_1-n_2)!n_2!} \cdots \frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!n_r!}$$

choices in all. After cancellation, this yields the desired result. ■