EXAMPLE B A stick of unit length is broken randomly in two places. What is the average length of the middle piece?

We interpret this question to mean that the locations of the two break points are independent uniform random variables U_1 and U_2. Therefore, we need to compute $E|U_1 - U_2|$. Theorem B tells us that we do not need to find the density function of $|U_1 - U_2|$ and that we can just integrate $|u_1 - u_2|$ against the joint density of U_1 and U_2, $f(u_1, u_2) = 1, 0 \le u_1 \le 1, 0 \le u_2 \le 1$. Thus,

$$
\begin{aligned}
E|U_1 - U_2| &= \int_0^1 \int_0^1 |u_1 - u_2| \, du_1 \, du_2 \\
&= \int_0^1 \int_0^{u_1} (u_1 - u_2) \, du_2 \, du_1 + \int_0^1 \int_{u_1}^1 (u_2 - u_1) \, du_2 \, du_1
\end{aligned}
$$

With some care, we find the expectation to be $\frac{1}{3}$. This is in accord with the intuitive argument that the smaller of U_1 and U_2 should be $\frac{1}{3}$ on the average and the larger should be $\frac{2}{3}$ on the average, which means that the average difference should be $\frac{1}{3}$. ■

We note the following immediate consequence of Theorem B.

COROLLARY A

If X and Y are independent random variables and g and h are fixed functions, then $E[g(X)h(Y)] = \{E[g(X)]\}\{E[h(Y)]\}$, provided that the expectations on the right-hand side exist. ■

In particular, if X and Y are independent, $E(XY) = E(X)E(Y)$. The proof of this corollary is left to Problem 29 of the end-of-chapter problems.

4.1.2 Expectations of Linear Combinations of Random Variables

One of the most useful properties of the expectation is that it is a linear operation. Suppose that you were told that the average temperature on July 1 in a certain location was 70°F, and you were asked what the average temperature in degrees Celsius was. You can simply convert to degrees Celsius and obtain $\frac{5}{9} \times 70 - 17.7 = 21.2$°C. The notion of the average value of a random variable, which we have defined as the expected value of a random variable, behaves in the same fashion. If $Y = aX + b$, then $E(Y) = aE(X) + b$. More generally, this property extends to linear combinations of random variables.