This integral is easy to evaluate once we realize that $\lambda^{\alpha+1} x^{\alpha} e^{-\lambda x} / \Gamma(\alpha+1)$ is a gamma density and therefore integrates to 1. We thus have

$$\int_0^\infty x^\alpha e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$$

from which it follows that

$$E(X) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \left[\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \right]$$

Finally, using the relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we find

$$E(X) = \frac{\alpha}{\lambda}$$

For the exponential density, $\alpha = 1$, so $E(X) = 1/\lambda$. This may be contrasted to the median of the exponential density, which was found in Section 2.2.1 to be $\log 2/\lambda$. The mean and the median can both be interpreted as "typical" values of X, but they measure different attributes of the probability distribution. ■

EXAMPLE **F** *Normal Distribution*

From the definition of the expectation, we have

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx$$

Making the change of variables $z = x - \mu$ changes this equation to

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2\sigma^2} dz + \frac{\mu}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2} dz$$

The first integral is 0 since the contributions from $z < 0$ cancel those from $z > 0$, and the second integral is μ because the normal density integrates to 1. Thus,

$$E(X) = \mu$$

The parameter μ of the normal density is the expectation, or mean value. We could have made the derivation much shorter by claiming that it was "obvious" that since the center of symmetry of the density is μ, the expectation must be μ. ■

EXAMPLE **G** *Cauchy Density*

Recall that the Cauchy density is

$$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right), \qquad -\infty < x < \infty$$

The density is symmetric about zero, so it would seem that $E(X) = 0$. However,

$$\int_{-\infty}^{\infty} \frac{|x|}{1+x^2} = \infty$$