# Lecture 17

*Lecture date: Oct 27* *Scribe: Sebastien Roch*

Our main result in this lecture is a proof that a function with low influence sum is "simple"—i.e. it depends on a small number of variables. This uses hypercontractivity results from previous lectures.

## 1 Low-Influence-Sum Functions Depend on Few Coordinates

The following theorem generalizes Friedgut [Fr98].

**Theorem 1** *Let $\mu$ be a probability distribution where the smallest atom has probability $\alpha$. Let $f \in L^2(\mu^n)$ be $\pm 1$-valued with*

$$I(f) = \sum_{i=1}^{n} I_i(f) = \sum_{S} |S|\, |f_S|_2^2 \leq b. \tag{1}$$

*Then there exists a function $g$, $\pm 1$-valued, satisfying*

$$\mathbf{P}[f \neq g] \leq \epsilon,$$

*such that $g$ depends on at most*

$$\mathcal{C} = \frac{b^2 2^{12b/\epsilon} + 3}{\epsilon^3 \alpha^{2b/\epsilon+2}}$$

*coordinates. Note that $\mathcal{C}$ does not depend on $n$.*

The proof requires the following straightforward lemma.

**Lemma 2** *Let $1 \leq q \leq 2$ and suppose that*

$$\mathbf{P}\left[|f| \in \{0\} \cup [\lambda, +\infty)\right] = 1,$$

*where $\lambda > 0$, then*

$$|f|_q^q \leq \lambda^{q-2} |f|_2^2.$$

**Proof:** (Lemma) It is easy to check that the inequality holds pointwise. The result follows. □

We now proceed with the proof of the theorem.

**Proof:** (Theorem) Let $\gamma > 0$ and

$$J = \{i \,:\, I_i(f) < \gamma\}.$$

Our goal is to show that, for a well chosen value of $\gamma$, the set $J$ is such that if

$$h = \sum_{S : S \cap J = \emptyset} f_S$$

then

$$\mathbf{P}[f \neq \mathrm{sgn}(h)] \leq \epsilon.$$

For this, it suffices to show that

$$|h - f|_2^2 \leq \epsilon.$$

Let

$$f_i = \sum_{S : i \in S} f_S.$$

Note first that $|f_i|_2^2 = I_i(f)$. Also, it is easy to see that for all $x$

$$f_i(x) = f(x) - \mathbf{E}[f \mid x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n],$$

(by expanding both sides over $\{f_S\}_{S \subseteq [n]}$). Therefore, it holds that for all $x$

$$\text{either } |f_i(x)| \geq 2\alpha \text{ or } f_i(x) = 0. \tag{2}$$

Let $q, \eta$ be such that the space $L^2(\mu^n)$ is $(q, 2, \eta)$-hypercontractive (we give actual constants below). Then, by Lemma 2, (2), and the assumptions of the Theorem, we have

$$\begin{aligned}
\sum_S |S \cap J| \, \eta^{2|S|} \, |f_S|_2^2 &= \sum_{i \in J} |T_\eta f_i|_2^2 \\
&\leq \sum_{i \in J} |f_i|_q^2 \\
&\leq \sum_{i \in J} \left((2\alpha)^{q-2} |f_i|_2^2\right)^{2/q} \\
&= (2\alpha)^{2-4/q} \sum_{i \in J} \left(|f_i|_2^2\right)^{2/q} \\
&\leq (2\alpha)^{2-4/q} \gamma^{2/q-1} \sum_i |f_i|_2^2 \\
&\leq (2\alpha)^{2-4/q} \gamma^{2/q-1} b \\
&\equiv t.
\end{aligned}$$

From this we get that

$$\sum \left\{ |f_S|_2^2 \,:\, |S \cap J| \geq \frac{2t\eta^{-2|S|}}{\epsilon} \right\} \leq \frac{\epsilon}{2}. \tag{3}$$

Observe furthermore that it follows from (1) that

$$\sum \left\{ |f_S|_2^2 \,:\, |S| \geq \frac{2b}{\epsilon} \right\} \leq \frac{\epsilon}{2}. \tag{4}$$

Combining (4) and (3), we obtain

$$\sum \left\{ |f_S|_2^2 \,:\, |S \cap J| \geq H \right\} \leq \epsilon,$$

where

$$H = \frac{2t\eta^{-4b/\epsilon}}{\epsilon}.$$

It remains to show that $H < 1$. For this, choose

$$q = \frac{3}{2}, \qquad \eta = \frac{\alpha^{1/6}}{2},$$

and

$$\gamma < \frac{\epsilon^3 \alpha^{2b/\epsilon+2}}{b^3 2^{12b/\epsilon+3}} = \alpha^{O(b/\epsilon)}\epsilon^3.$$

In particular, we have

$$|J^c| \leq \frac{b}{\gamma}.$$

□

**Tight Example.** The *tribes functions* provide a tight example for Theorem 1. Assume $n = T\,2^T$ for some $T > 0$. A tribe function is a function $f : \{-1,1\}_0^n \to \{-1,1\}$ defined as follows: think of the $n$ variables as being $2^T$ "tribes" of $T$ variables; a tribe has value $-1$ unless all tribe variables are 1; $f = 1$ iff there exists a tribe whose value is 1. It is easy to see that the probability that $f$ is 1 is roughly $1 - 1/e$ (Poisson variable). Also

$$I_i(f) \leq \mathbf{P}[\text{all other tribe variables are } 1] \leq 2^{-T+1} = O\left(\frac{\log n}{n}\right).$$

Therefore the influence sum is $O(\log n)$. The tightness is left as an exercise.

**Exercise 3** *(1 pt) Use tribes functions to show that Theorem 1 is tight, i.e. show that for tribes functions, $2^{\Omega(b/\epsilon)}$ coordinates are necessary to get an $\epsilon$-approximation.*

## 2 Dictators Grow Slowly

In this section, we show that among all monotone functions, the dictator functions are the ones that grow the slowest.

**Proposition 4** *Let $f : \{0,1\}^n \to \{0,1\}$ be a monotone function such that*

$$\frac{\mathbf{E}_p[f]}{1-\mathbf{E}_p[f]} = c\frac{p}{1-p},$$

*for some $c > 0$. Then if $q > p$,*

$$\frac{\mathbf{E}_q[f]}{1-\mathbf{E}_q[f]} \geq c\frac{q}{1-q}.$$

*This inequality is satisfied with equality for dictator functions.*

**Proof:** By Russo's formula,

$$\frac{d}{dt}\mathbf{E}^t[f] = \frac{\sum I_i^t(f)}{t(1-t)} \geq \frac{\mathbf{Var}^t[f]}{t(1-t)} = \frac{\mathbf{E}^t(f)(1-\mathbf{E}^t(f))}{t(1-t)}.$$

Solving this differential equation gives immediately

$$\frac{\mathbf{E}_t[f]}{1-\mathbf{E}_t[f]}\frac{1-\mathbf{E}_p[f]}{\mathbf{E}_p[f]} \geq \frac{t}{1-t}\frac{1-p}{p}.$$

□

## References

[Fr98] E. Friedgut, Boolean Functions With Low Average Sensitivity Depend On Few Coordinates, Combinatorica, 18(1):27–35, 1998.