

Problem Set 1

Fall 2012

Issued: Tuesday, August 28, 2012 **Due:** Tuesday, September 4, 2012

Reading: This homework is purely undergraduate review on probability and linear algebra; relevant material is covered in recitation on Wednesday, August 29th. If you are not familiar with the concepts here, then you *do not have the appropriate background for this course*, and will find the course too demanding. In this case, it would be best to drop the class, which would allow someone on the wait list with the appropriate background to enroll. This homework must be done alone, without consulting any classmates or friends.

Problem 1.1

Consider the collection of vectors:

$$[1 \ 1 \ 1 \ x], \quad [1 \ 1 \ x \ 1], \quad [1 \ x \ 1 \ 1], \quad [x \ 1 \ 1 \ 1].$$

- (a) For which real numbers x do these vectors *not* form a basis of \mathbb{R}^4 ?
- (b) For each value of x from (a), what is the dimension of the subspace of \mathbb{R}^4 that they span?

Problem 1.2

For square matrices, prove the following properties of the matrix trace and determinant:

- (a) $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$.
- (b) $\text{trace}(AB) = \text{trace}(BA)$.
- (c) $\det(AB) = \det(A)\det(B)$.
- (d) For a non-singular matrix, $\det(A^{-1}) = 1/\det(A)$.

Problem 1.3

Consider the $(k + m)$ -dimensional matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathbb{R}^{k \times k}$, $D \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{k \times m}$ and $C \in \mathbb{R}^{m \times k}$.

- (a) If $B = 0$ and $C = 0$, show that $\det(M) = \det(A) \det(D)$.
- (b) If A is invertible, show that $\det(M) = \det(A) \det(D - CA^{-1}B)$.

Problem 1.4

Prove or disprove: a symmetric matrix A is positive semidefinite if and only if $\text{trace}(AB) \geq 0$ for all symmetric positive semidefinite matrices B .

Problem 1.5

Let $X = (X_1, \dots, X_n)$ be a jointly Gaussian random vector with mean $\mu \in \mathbb{R}^n$ and covariance matrix Σ . Let $W = (W_1, \dots, W_n)$ be a second jointly Gaussian random vector with mean ν and covariance Λ .

- (a) Let A and B be $n \times n$ matrices, and form the random vector $Y = AX + BW$. Compute the mean vector and covariance matrix of Y . (Your answer can involve $\text{cov}(X, W)$.)
- (b) How does your answer change if X and W are uncorrelated?

Problem 1.6

Craig is doing a study of moose in the Alaskan wilderness, and wants to estimate their heights. Let X be the height in meters of a randomly selected moose. Craig is interested in estimating $h = \mathbb{E}[X]$. Being sure that no moose is taller than 3 meters, Craig decides to use 1.5 meters as a conservative (large) value for the standard deviation of X . To estimate h , Craig compute the average H of the heights of n moose that he selects at random.

- (a) Compute $\mathbb{E}[H]$ and $\text{var}(H)$ in terms of h and Craig's 1.5 meter bound for $\text{std}(X)$.
- (b) Compute the minimum value of n (with $n > 0$) such that the standard deviation of H will be less than 0.01 meters.
- (c) Say Craig would like to be 99% sure that his estimate is within 5 centimeters of the true average height of moose. Using the Chebyshev inequality, calculate the minimum value of n required.
- (d) If we agree that no moose are taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for X , the height of any moose selected at random?

Problem 1.7

A group of N archers shoot at a target. The distance of each shot from the center of the target is uniformly distributed between 0 to 1, independently of the other shots.

- (a) Find the expected distance from the winner's arrow to the center. (The winner's arrow is closest to the origin.)
- (b) Find the expected distance from the loser's arrow to the center. (The loser's arrow is the arrow farthest away from the origin).

Problem 1.8

Every day that he leaves work, Fred the Absent-minded Accountant toggles his light switch according to the following protocol: (i) if the light is on, he switches it off with probability 0.60; and (ii) if the light is off, he switches it on with probability 0.20. At no other time (other than the end of each day) is the light switch touched.

- (a) Suppose that on Monday night after leaving work, Fred's office is equally likely to be light or dark. What is the probability that his office will be lit all five nights of the week (Monday through Friday)?
- (b) Suppose that you observe that his office is lit on both Monday and Friday nights after work. Compute the expected number of nights, from that Monday through Friday, that his office is lit.
- (c) Suppose that Fred's office is lit on Monday night after work. Compute the expected number of days until the first night that his office is dark.

Now suppose that Fred has been working for five years (i.e., assume that the Markov chain is in steady state).

- (d) Is his light more likely to be on or off at the end of a given workday?