Some general theory for weighted regressions March 16/08

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We use the setup for Section 7 in Freedman and Berk (2008), with only one period. We have the unweighted world governed by $\pi$, and the weighted world governed by $\tilde{\pi}$. In the unweighted world,

(i) subjects are IID,
(ii) $Z, V$ are independent,
(iii) $\pi(X = 1|Z, V) = p(Z)$, where $0 < p(Z) < 1$ does not depend on $V$,
(iv) $Y = \Phi(X, Z, V)$, where $\Phi$ is a measurable function that does not depend on $U$,
(v) $V$ is independent of $(X, Z)$.

Conditions (ii)–(iv) apply the common (i.e., population-level) distribution of the subjects, and (v) follows from (ii)–(iv).

We now reweight to $\pi$ to the probability $\tilde{\pi}$:

$$\begin{aligned}\frac{d\tilde{\pi}}{d\pi} &= \frac{c}{p(Z)} \text{ on } \{X = 1\}\\ &= \frac{c}{1 - p(Z)} \text{ on } \{X = 0\}\end{aligned}$$

As Lemma 1 shows, $c$ must be $1/2$ in order for $\tilde{\pi}$ to be a probability; properties (i)–(ii) and (iv)–(v) are preserved, but (iii) becomes

$$\tilde{\pi}(X = 1|Z, V) = 1/2 \qquad (*)$$

Furthermore, the joint distribution of $Z, V$ is unchanged; so is the relationship between $Y$ and $X, Z, V$. All that changes is the conditional distribution of $X$ given $Z$.

Lemma 1. Let $X$ and $W$ be random variables on $(\Omega, \mathcal{F}, \pi)$, with $X = 0$ or $1$ while $W$ takes values in a complete separable metric space $\mathcal{M}$. Suppose $\pi(X = 1|W) = p(W)$ where $0 < p < 1$ is a Borel function on $\mathcal{M}$. Let

$$\phi = \frac{X}{p(W)} + \frac{1 - X}{1 - p(W)},$$

a finite, positive, $\mathcal{F}$-measurable function on $\Omega$. Let $c$ be a positive real number. Define the $\sigma$-finite measure $\tilde{\pi}$ on $(\Omega, \mathcal{F})$ by

$$\frac{d\tilde{\pi}}{d\pi} = c\phi.$$

Let $B$ be a Borel subset of $\mathcal{M}$. Then—

(i) $\tilde{\pi}(X = 1 \,\&\, W \in B) = c\pi(W \in B)$.
(ii) $\tilde{\pi}(X = 0 \,\&\, W \in B) = c\pi(W \in B)$.
(iii) $\tilde{\pi}(\Omega) = 2c$.
(iv) $\tilde{\pi}$ is a probability measure iff $c = 1/2$.
(v) If $c = 1/2$, the $\tilde{\pi}$-distribution of $W$ coincides with the $\pi$-distribution.
(vi) If $c = 1/2$, then $\tilde{\pi}(X = 1|W) = 1/2$.

Proof. Write $1_B$ for the indicator function of $B$. This is a Borel function on $\mathcal{M}$. Then

$$\begin{aligned}
\tilde{\pi}(X = 1 \,\&\, W \in B) &= \int_{\{X=1\}} 1_B(W)\, d\tilde{\pi} \\
&= \int_{\{X=1\}} 1_B(W) \frac{d\tilde{\pi}}{d\pi}\, d\pi \\
&= c \int_{\{X=1\}} \frac{1}{p(W)} 1_B(W)\, d\pi \\
&= c E_\pi \Big[ \frac{X}{p(W)} 1_B(W) \Big] \\
&= c E_\pi \left\{ E_\pi \Big[ \frac{X}{p(W)} 1_B(W) \Big| W \Big] \right\} \qquad (**) \\
&= c E_\pi \left\{ \frac{p(W)}{p(W)} 1_B(W) \right\} \\
&= c E_\pi \big\{ 1_B(W) \big\} \\
&= c\pi(W \in B)
\end{aligned}$$

because $E_\pi(X|W) = p(W)$ on the right hand side of $(**)$. This proves (i), and (ii) is similar. Then (iii) and (iv) are immediate: take $B = \mathcal{M}$. Now (v) follows by adding (i) and (ii). Finally, (vi) is immediate from (i). QED

Discussion. $\pi$ describes the original, unweighted world; $\tilde{\pi}$ describes the weighted world. $X$ is treatment status, while $W = (Z, V)$ is the vector of covariates and latents used to construct the response $Y$, which is computed from $X$ and $W$ in the weighted world using the same formula as in the unweighted world.

Conclusion (v) of the lemma shows that $Z$ and $V$ are independent in the weighted world; (vi) proves $(*)$, and hence the independence of $V$ from $(X, Z)$. We still have $Y = \Phi(X, Z, V)$, at least almost surely, because $\tilde{\pi} \equiv \pi$.

To be clearer (but fussier), we should start with $(X, Z, V, Y)$ defined on some probability triple $(\Omega, \mathcal{F}, \pi)$, impose conditions (ii)–(v), then define $\tilde{\pi}$ and prove the claims about it. After that, we could introduce IID copies of $(X, Z, V, Y)$. Each copy would be reweighted. For instance, we could simply take Cartesian products of the basic triple with itself. The unweighted world corresponds to $(\Omega, \mathcal{F}, \pi)^{\mathbb{Z}}$ and the weighted world is $(\Omega, \mathcal{F}, \tilde{\pi})^{\mathbb{Z}}$, where $\mathbb{Z}$ is the sequence of positive integers.

NB. The sample is blown up to population level using the weights, and sampling error is ignored. This is a one-period model, but the argument generalizes to several periods, as we discuss next.

Two periods

The setup is the same, except there are treatment variables $X_1$ and $X_2$ for each the two periods, with $\pi(X_1 = 1|W) = p_1(W)$ and $\pi(X_2 = 1|X_1, W) = p_2(X_1, W)$, these functions being positive and less than 1. Let

$$\phi = \frac{1}{p_1(W)} \times \frac{1}{p_2(1, W)} \quad \text{on} \quad \{X_1 = 1 \,\&\, X_2 = 1\}$$

$$= \frac{1}{p_1(W)} \times \frac{1}{1 - p_2(1, W)} \text{ on } \{X_1 = 1 \,\&\, X_2 = 0\}$$
$$= \frac{1}{1 - p_1(W)} \times \frac{1}{p_2(0, W)} \text{ on } \{X_1 = 0 \,\&\, X_2 = 1\}$$
$$= \frac{1}{1 - p_1(W)} \times \frac{1}{1 - p_2(0, W)} \text{ on } \{X_1 = 0 \,\&\, X_2 = 0\}$$

and let

$$\frac{d\tilde{\pi}}{d\pi} = c\phi.$$

As before, $\tilde{\pi}(X_1 = x_1 \,\&\, X_2 = x_2 \,\&\, W \in B) = c\pi(W \in B)$. For example, take $x_1 = x_2 = 1$. To simplify the analog of $(**)$ in the proof, we would compute

$$E_\pi\left[\frac{X_1 X_2 1_B(W)}{p_1(W)p_2(1, W)}\middle| W\right] = \frac{1_B(W)}{p_1(W)p_2(1, W)}\,\pi(X_1 = 1 \,\&\, X_2 = 1|W)$$
$$= \frac{1_B(W)}{p_1(W)p_2(1, W)}\,\pi(X_1 = 1|W)\pi(X_2 = 1|X_1 = 1, W)$$
$$= \frac{1_B(W)}{p_1(W)p_2(1, W)}\,p_1(W)p_2(1, W)$$
$$= 1_B(W).$$

Thus, $c = 1/4$ if $\tilde{\pi}$ is to be a probability, and the argument proceeds as before. In the weighted world, i.e., relative to $\tilde{\pi}$ with $c = 1/4$,

- the distribution of $W$ is unchanged,
- $X_1$, $X_2$, and $W$ are independent,
- $X_1$ and $X_2$ are each 0 or 1 with probability 1/2.

Here, $W$ represents the initial covariates, as well as the latents used to update covariates, select treatments, and compute responses. The responses $Y_1, Y_2$ would be computed from treatment variables and latents using the same formulas in the weighted and unweighted worlds, covariates would be updated the same way, etc. The extension to $n$ periods is straightforward.

Example. In Simulation #1 of Freedman and Berk (2008), let $Z_2 = \alpha + \beta Z_1 + \delta$, where $\delta$ is random, mean 0, independent of $Z_1$. If you omit $c_2 Z_2$ and run a weighted regression of $Y$ on $X$ and $Z_1$, then $\hat{a}$ estimates $a + c_2\alpha$. Given the parameter values in the simulation, namely, $\beta = 1/2$ and $\alpha = 3/4$, the estimand is 2.5, in accordance with the simulation results in Table 1 of that paper.

Reference

D. A. Freedman and R. A. Berk. "Weighting regressions by propensity scores." In press, *Evaluation Review*, vol. 32 (2008). http://www.stat.berkeley.edu/ census/weight.pdf