Introduction to Time Series Analysis. Lecture 9.

Peter Bartlett

Last lecture:

1. Forecasting and backcasting.
2. Prediction operator.
3. Partial autocorrelation function.

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1. Review: Forecasting
2. Partial autocorrelation function.
3. Recursive methods: Durbin-Levinson.
4. The innovations representation.
5. Recursive methods: Innovations algorithm.
6. Example: Innovations algorithm for forecasting an MA(1)

Review: One-step-ahead linear prediction

$$X_{n+1}^{n} = \phi_{n1}X_n + \phi_{n2}X_{n-1} + \cdots + \phi_{nn}X_1$$

$$\Gamma_n \phi_n = \gamma_n,$$

$$P_{n+1}^{n} = \mathrm{E}\left(X_{n+1} - X_{n+1}^{n}\right)^2 = \gamma(0) - \gamma_n' \Gamma_n^{-1}\gamma_n,$$

$$\Gamma_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & & \gamma(n-2) \\ \vdots & & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix},$$

$$\phi_n = (\phi_{n1}, \phi_{n2}, \ldots, \phi_{nn})', \quad \gamma_n = (\gamma(1), \gamma(2), \ldots, \gamma(n))'.$$

Review: The prediction operator

For random variables $Y, Z_1, \ldots, Z_n$, define the **best linear prediction of** Y **given** $Z = (Z_1, \ldots, Z_n)'$ as the operator $P(\cdot|Z)$ applied to Y:

$$P(Y|Z) = \mu_Y + \phi'(Z - \mu_Z)$$

with

$$\Gamma\phi = \gamma,$$

where

$$\gamma = \text{Cov}(Y, Z)$$

$$\Gamma = \text{Cov}(Z, Z).$$

Review: Properties of the prediction operator

1. $\mathrm{E}(Y - P(Y|Z)) = 0$, $\mathrm{E}((Y - P(Y|Z))Z) = 0$.

2. $\mathrm{E}((Y - P(Y|Z))^2) = \mathrm{Var}(Y) - \phi'\gamma$.

3. $P(\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_0 | Z) = \alpha_0 + \alpha_1 P(Y_1|Z) + \alpha_2 P(Y_2|Z)$.

4. $P(Z_i|Z) = Z_i$.

5. $P(Y|Z) = \mathrm{E}Y$ if $\gamma = 0$.

Review: Partial autocorrelation function

The Partial AutoCorrelation Function (PACF) of a stationary time series $\{X_t\}$ is

$$\phi_{11} = \text{Corr}(X_1, X_0) = \rho(1)$$

$$\phi_{hh} = \text{Corr}(X_h - X_h^{h-1}, X_0 - X_0^{h-1}) \quad \text{for } h = 2, 3, \ldots$$

This removes the linear effects of $X_1, \ldots, X_{h-1}$:

$$\ldots, X_{-1}, \underline{X_0}, \underbrace{X_1, X_2, \ldots, X_{h-1}}_{\text{partial out}}, \underline{X_h}, X_{h+1}, \ldots$$

Review: Partial autocorrelation function

The PACF ϕ_{hh} is also the last coefficient in the best linear prediction of X_{h+1} given $X_1, \ldots, X_h$:

$$\Gamma_h \phi_h = \gamma_h \qquad X_{h+1}^h = \phi_h' X$$

$$\phi_h = (\phi_{h1}, \phi_{h2}, \ldots, \phi_{hh}).$$

Example: PACF of an AR(p)

$$\text{For} \quad X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + W_t,$$

$$X_{n+1}^n = \sum_{i=1}^{p} \phi_i X_{n+1-i}.$$

$$\text{Thus, } \phi_{hh} = \begin{cases} \phi_h & \text{if } 1 \le h \le p \\ 0 & \text{otherwise.} \end{cases}$$

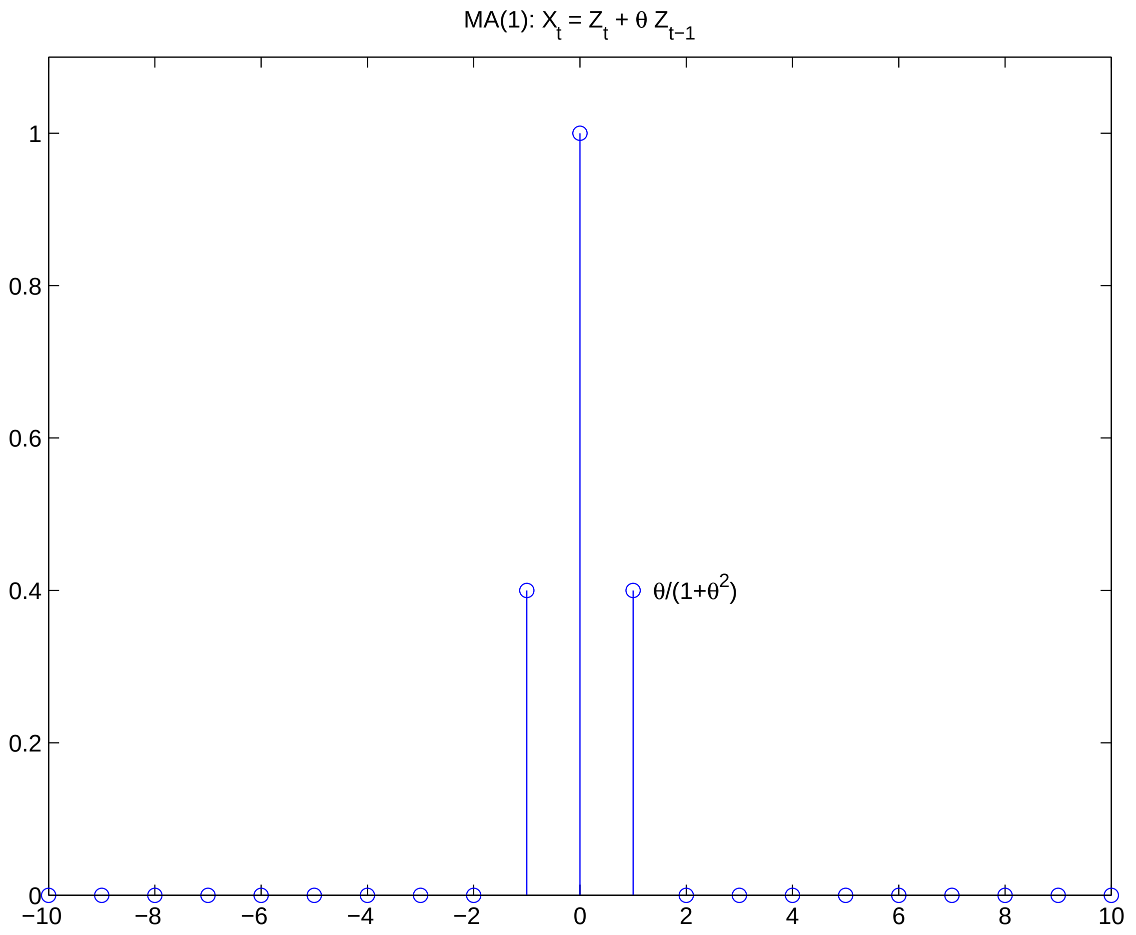
Example: PACF of an invertible MA(q)

For $X_t = \sum_{i=1}^{q} \theta_i W_{t-i} + W_t, \qquad X_t = -\sum_{i=1}^{\infty} \pi_i X_{t-i} + W_t,$

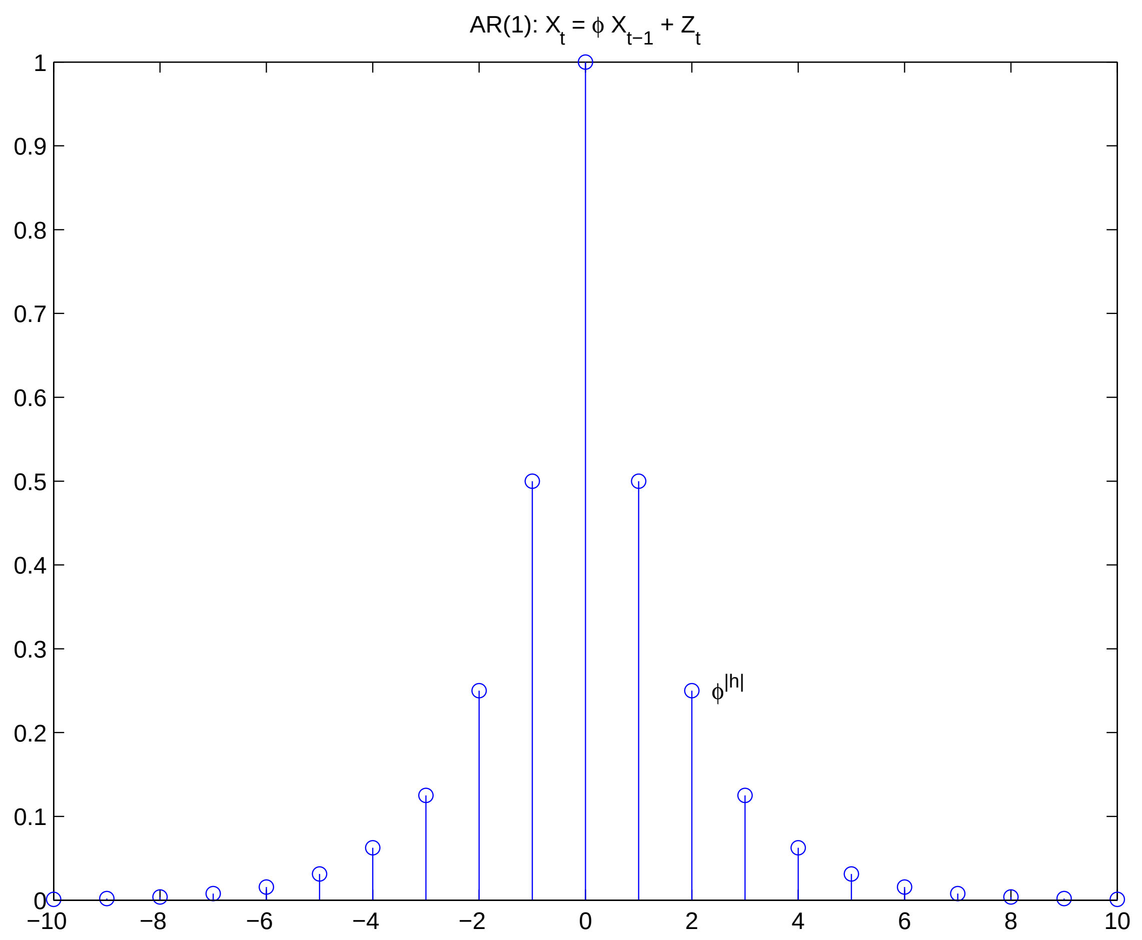
$$\begin{aligned}
X_{n+1}^n &= P(X_{n+1}|X_1, \ldots, X_n) \\
&= P\left(-\sum_{i=1}^{\infty} \pi_i X_{n+1-i} + W_{n+1}|X_1, \ldots, X_n\right) \\
&= -\sum_{i=1}^{\infty} \pi_i P\left(X_{n+1-i}|X_1, \ldots, X_n\right) \\
&= -\sum_{i=1}^{n} \pi_i X_{n+1-i} - \sum_{i=n+1}^{\infty} \pi_i P\left(X_{n+1-i}|X_1, \ldots, X_n\right).
\end{aligned}$$

In general, $\phi_{hh} \neq 0$.

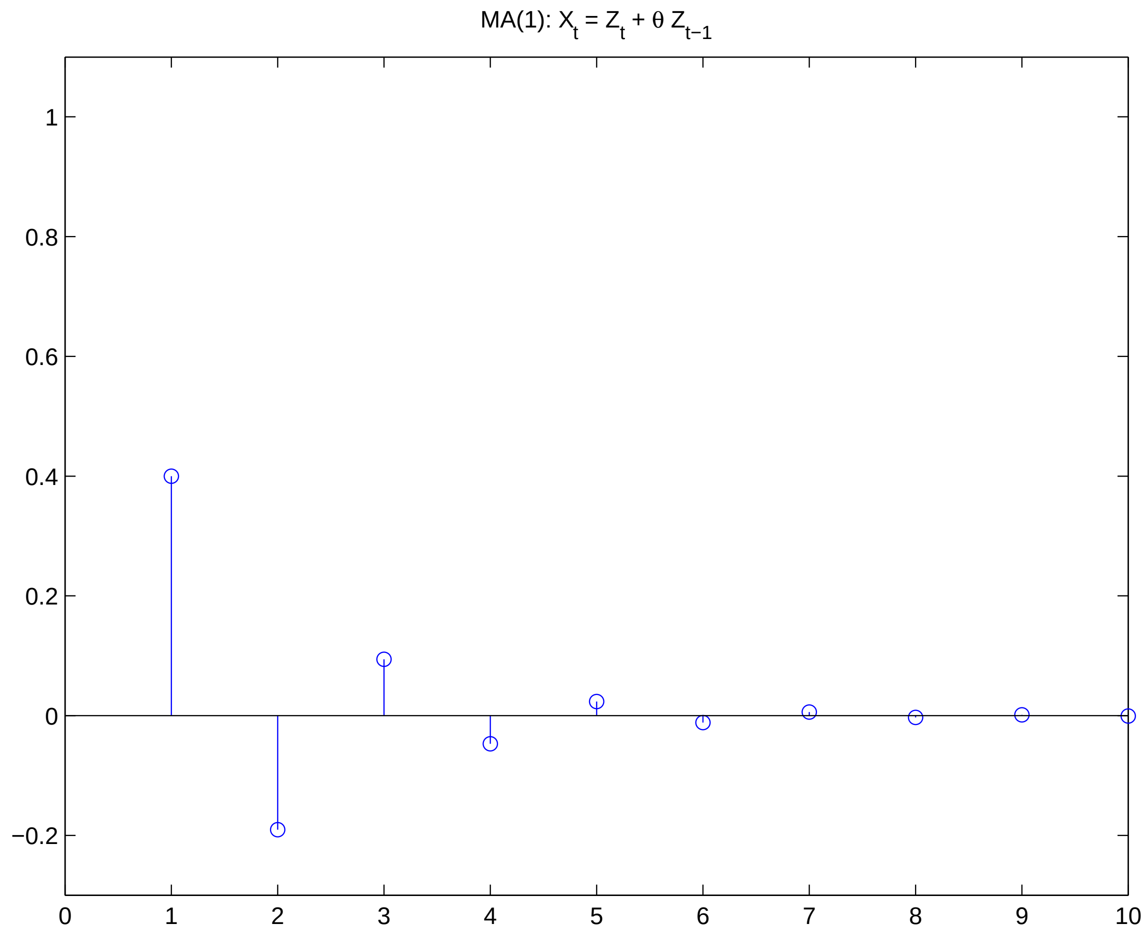
ACF of the MA(1) process



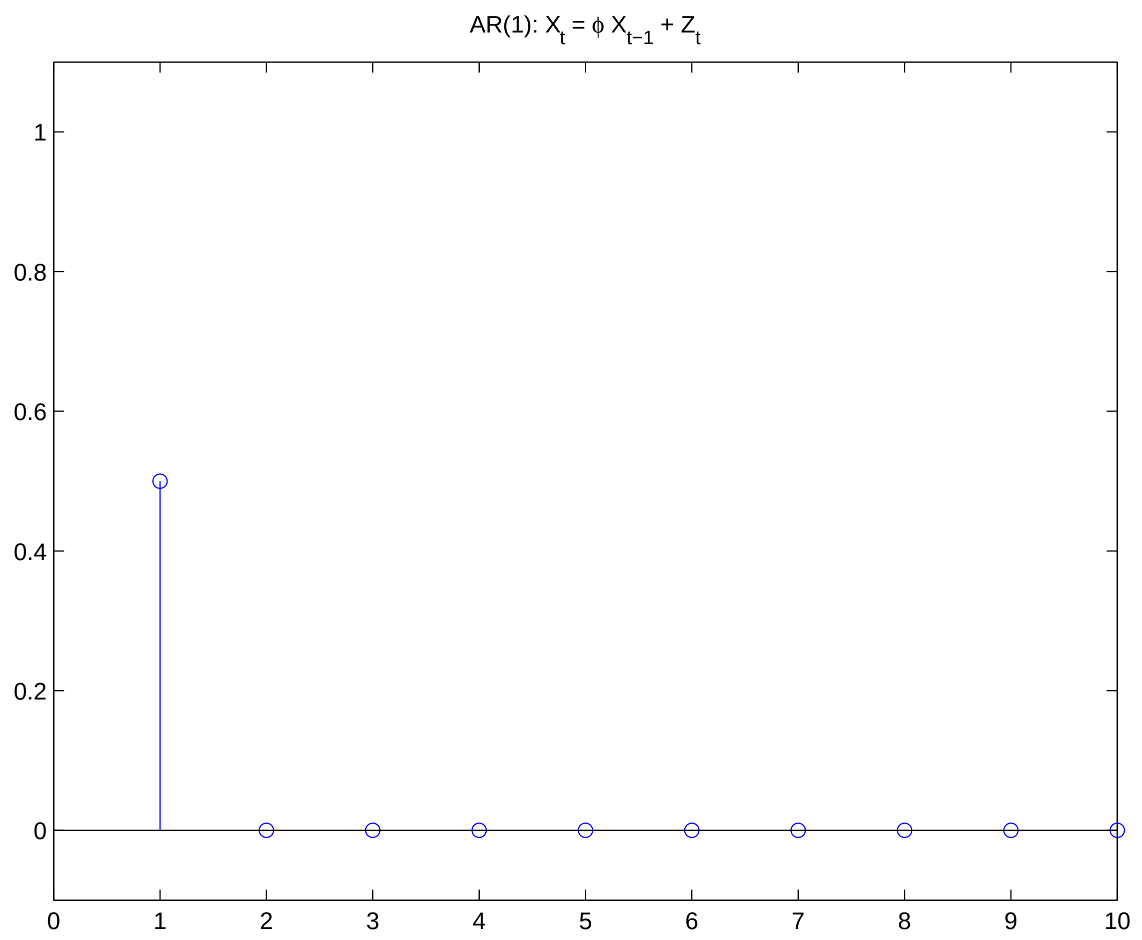
ACF of the AR(1) process



PACF of the MA(1) process



PACF of the AR(1) process



PACF and ACF

Model:	**ACF:**	**PACF:**
AR(p)	decays	zero for $h > p$
MA(q)	zero for $h > q$	decays
ARMA(p,q)	decays	decays

Sample PACF

For a realization $x_1, \ldots, x_n$ of a time series, the **sample PACF** is defined by

$$\hat{\phi}_{00} = 1$$

$$\hat{\phi}_{hh} = \text{ last component of } \hat{\phi}_h,$$

where $\hat{\phi}_h = \hat{\Gamma}_h^{-1} \hat{\gamma}_h$.

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The importance of P_{n+1}^n: Prediction intervals

$$X_{n+1}^n = \phi_{n1} X_n + \phi_{n2} X_{n-1} + \cdots + \phi_{nn} X_1$$

$$\Gamma_n \phi_n = \gamma_n, \qquad P_{n+1}^n = \mathrm{E}\left(X_{n+1} - X_{n+1}^n\right)^2 = \gamma(0) - \gamma_n' \Gamma_n^{-1} \gamma_n.$$

After seeing $X_1, \ldots, X_n$, we forecast X_{n+1}^n. The expected squared error of our forecast is P_{n+1}^n. We can construct a prediction interval:

$$X_{n+1}^n \pm c_{\alpha/2} \sqrt{P_{n+1}^n}.$$

For a Gaussian process, the prediction error has distribution $\mathcal{N}(0, P_{n+1}^n)$, so $c_{0.05/2} = 1.96$ gives a 95% prediction interval.

Computing linear prediction coefficients

$$X_{n+1}^n = \phi_{n1} X_n + \phi_{n2} X_{n-1} + \cdots + \phi_{nn} X_1$$

$$\Gamma_n \phi_n = \gamma_n,$$

$$P_{n+1}^n = \mathrm{E}\left(X_{n+1} - X_{n+1}^n\right)^2 = \gamma(0) - \gamma_n' \Gamma_n^{-1} \gamma_n.$$

How can we compute these quantities recursively?
i.e., given the coefficients ϕ_{n-1} of X_n^{n-1}, how can we compute the coefficients ϕ_n of X_{n+1}^n, without solving another linear system $\Gamma_n \phi_n = \gamma_n$?

Durbin-Levinson

$$\phi_0 = 0, \qquad \phi_{00} = 0;$$

$$\phi_1 = \phi_{11}, \qquad \phi_{11} = \frac{\gamma(1)}{\gamma(0)};$$

$$\phi_n = \begin{pmatrix} \phi_{n-1} - \phi_{nn}\tilde{\phi}_{n-1} \\ \phi_{nn} \end{pmatrix}, \quad \phi_{nn} = \frac{\gamma(n) - \phi'_{n-1}\tilde{\gamma}_{n-1}}{\gamma(0) - \phi'_{n-1}\gamma_{n-1}}.$$

$$\phi_n = (\phi_{n1}, \ldots, \phi_{nn})' \qquad \tilde{\phi}_n = (\phi_{nn}, \ldots, \phi_{n1})',$$

$$\gamma_n = (\gamma(1), \ldots, \gamma(n))' \qquad \tilde{\gamma}_n = (\gamma(n), \ldots, \gamma(1))'.$$

Durbin-Levinson: Example

$$\phi_0 = 0, \qquad \phi_{00} = 0;$$

$$\phi_1 = \phi_{11}, \qquad \phi_{11} = \frac{\gamma(1)}{\gamma(0)};$$

$$\phi_n = \begin{pmatrix} \phi_{n-1} - \phi_{nn}\tilde{\phi}_{n-1} \\ \phi_{nn} \end{pmatrix}, \quad \phi_{nn} = \frac{\gamma(n) - \phi'_{n-1}\tilde{\gamma}_{n-1}}{\gamma(0) - \phi'_{n-1}\gamma_{n-1}}.$$

This algorithm computes $\phi_1, \phi_2, \phi_3, \ldots$, where

$$X_2^1 = X_1\phi_1, \quad X_3^2 = (X_2, X_1)\phi_2, \quad X_4^3 = (X_3, X_2, X_1)\phi_3, \ldots$$

Durbin-Levinson: Example

$$\phi_1 = \phi_{11}, \qquad \phi_{11} = \frac{\gamma(1)}{\gamma(0)};$$

$$\phi_n = \begin{pmatrix} \phi_{n-1} - \phi_{nn}\tilde{\phi}_{n-1} \\ \phi_{nn} \end{pmatrix}, \quad \phi_{nn} = \frac{\gamma(n) - \phi'_{n-1}\tilde{\gamma}_{n-1}}{\gamma(0) - \phi'_{n-1}\gamma_{n-1}}.$$

$$\phi_1 = \gamma(1)/\gamma(0),$$

$$\phi_2 = \begin{pmatrix} \phi_1 - \phi_{22}\phi_{11} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} \frac{\gamma(1)}{\gamma(0)}\left(1 - \frac{\gamma(2)-\gamma(1)}{\gamma(0)-\gamma(1)}\right) \\ \frac{\gamma(2)-\gamma(1)}{\gamma(0)-\gamma(1)} \end{pmatrix}, \text{ etc.}$$

Durbin-Levinson: Why it works (Details)

Clearly, $\Gamma_1 \phi_1 = \gamma_1$.

Suppose $\Gamma_{n-1}\phi_{n-1} = \gamma_{n-1}$. Then $\Gamma_{n-1}\tilde{\phi}_{n-1} = \tilde{\gamma}_{n-1}$, and so

$$\Gamma_n \phi_n = \begin{pmatrix} \Gamma_{n-1} & \tilde{\gamma}_{n-1} \\ \tilde{\gamma}'_{n-1} & \gamma(0) \end{pmatrix} \begin{pmatrix} \phi_{n-1} - \phi_{nn}\tilde{\phi}_{n-1} \\ \phi_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{n-1} \\ \tilde{\gamma}'_{n-1}\phi_{n-1} + \phi_{nn}\left(\gamma(0) - \gamma'_{n-1}\phi_{n-1}\right) \end{pmatrix}$$

$$= \gamma_n.$$

Durbin-Levinson: Evolution of mean square error

$$
\begin{aligned}
P^n_{n+1} &= \gamma(0) - \phi_n' \gamma_n \\
&= \gamma(0) - \begin{pmatrix} \phi_{n-1} - \phi_{nn}\tilde{\phi}_{n-1} \\ \phi_{nn} \end{pmatrix}' \begin{pmatrix} \gamma_{n-1} \\ \gamma(n) \end{pmatrix} \\
&= P^{n-1}_n - \phi_{nn}\left(\gamma(n) - \tilde{\phi}'_{n-1}\gamma_{n-1}\right) \\
&= P^{n-1}_n - \phi^2_{nn}\left(\gamma(0) - \phi'_{n-1}\gamma_{n-1}\right) \qquad \text{(From expression for } \phi_{nn}\text{)} \\
&= P^{n-1}_n\left(1 - \phi^2_{nn}\right).
\end{aligned}
$$

i.e., variance reduces by a factor $1 - \phi^2_{nn}$.

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The innovations representation

Instead of writing the best linear predictor as

$$X_{n+1}^n = \phi_{n1} X_n + \phi_{n2} X_{n-1} + \cdots + \phi_{nn} X_1,$$

we can write

$$X_{n+1}^n = \theta_{n1} \underbrace{\left(X_n - X_n^{n-1}\right)}_{\text{innovation}} + \theta_{n2} \left(X_{n-1} - X_{n-1}^{n-2}\right) + \cdots + \theta_{nn} \left(X_1 - X_1^0\right).$$

This is still linear in $X_1, \ldots, X_n$.

The innovations are uncorrelated:
$\mathrm{Cov}(X_j - X_j^{j-1}, X_i - X_i^{i-1}) = 0$ for $i \neq j$.

Comparing representations: $U_n = X_n - X_n^{n-1}$ versus X_n

$\{U_n\}$ form a *decorrelated* representation for the $\{X_n\}$:

$$\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\phi_{11} & 1 & & 0 \\ \vdots & & \ddots & \\ -\phi_{n-1,n-1} & -\phi_{n-1,n-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

Comparing representations: $U_n = X_n - X_n^{n-1}$ versus X_n

$$\begin{pmatrix} X_1^0 \\ X_2^1 \\ \vdots \\ X_n^{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \theta_{11} & 0 & & 0 \\ \vdots & & \ddots & \\ \theta_{n-1,n-1} & \theta_{n-1,n-2} & \cdots & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$$

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Innovations Algorithm

$$X_1^0 = 0, \qquad X_{n+1}^n = \sum_{i=1}^{n} \theta_{ni} \left(X_{n+1-i} - X_{n+1-i}^{n-i} \right).$$

$$\theta_{n,n-i} = \frac{1}{P_{i+1}^i} \left(\gamma(n-i) - \sum_{j=0}^{i-1} \theta_{i,i-j} \theta_{n,n-j} P_{j+1}^j \right).$$

$$P_1^0 = \gamma(0) \qquad P_{n+1}^n = \gamma(0) - \sum_{i=0}^{n-1} \theta_{n,n-i}^2 P_{i+1}^i.$$

Innovations Algorithm: Example

$$\theta_{n,n-i} = \frac{1}{P_{i+1}^{i}} \left(\gamma(n-i) - \sum_{j=0}^{i-1} \theta_{i,i-j} \theta_{n,n-j} P_{j+1}^{j} \right).$$

$$P_1^0 = \gamma(0) \qquad P_{n+1}^{n} = \gamma(0) - \sum_{i=0}^{n-1} \theta_{n,n-i}^2 P_{i+1}^{i}.$$

$$\theta_{1,1} = \gamma(1)/P_1^0, \qquad P_2^1 = \gamma(0) - \theta_{1,1}^2 P_1^0$$

$$\theta_{2,2} = \gamma(2)/P_1^0, \quad \theta_{2,1} = \left(\gamma(1) - \theta_{1,1}\theta_{2,2}P_1^0\right)/P_2^1,$$

$$P_3^2 = \gamma(0) - \left(\theta_{2,2}^2 P_1^0 + \theta_{2,1}^2 P_2^1\right)$$

$$\theta_{3,3}, \quad \theta_{3,2}, \quad \theta_{3,1}, \quad P_4^3, \ldots$$

Predicting h steps ahead using innovations

The innovations representation for the one-step-ahead forecast is

$$P(X_{n+1}|X_1, \ldots, X_n) = \sum_{i=1}^{n} \theta_{ni} \left(X_{n+1-i} - X_{n+1-i}^{n-i}\right),$$

What is the innovations representation for $P(X_{n+h}|X_1, \ldots, X_n)$?

It is $P(X_{n+h}|X_1, \ldots, X_{n+h-1})$, but with the unobserved innovations (from $n + 1$ to $n + h - 1$) set to zero.

Predicting h steps ahead using innovations

What is the innovations representation for $P(X_{n+h}|X_1, \ldots, X_n)$?

Fact: If $h \geq 1$ and $1 \leq i \leq n$, we have
$\mathrm{Cov}(X_{n+h} - P(X_{n+h}|X_1, \ldots, X_{n+h-1}), X_i) = 0.$

Thus, $P(X_{n+h} - P(X_{n+h}|X_1, \ldots, X_{n+h-1})|X_1, \ldots, X_n) = 0.$
That is, the best prediction of X_{n+h} is the
best prediction of the one-step-ahead forecast of X_{n+h}.

Fact: The best prediction of $X_{n+1} - X_{n+1}^n$ given only $X_1, \ldots, X_n$ is 0.

Similarly for $n+2, \ldots, n+h-1$.

Predicting h steps ahead using innovations

Innovations representation:

$$P(X_{n+h}|X_1,\ldots,X_n) = \sum_{i=1}^{n} \theta_{n+h-1,h-1+i} \left(X_{n+1-i} - X_{n+1-i}^{n-i}\right)$$

Predicting h steps ahead using innovations (Details)

$$
\begin{aligned}
&P(X_{n+h}|X_1,\ldots,X_n) \\
&= P\left(P(X_{n+h}|X_1,\ldots,X_{n+h-1})|X_1,\ldots,X_n\right) \\
&= P\left(\sum_{i=1}^{n+h-1} \theta_{n+h-1,i}\left(X_{n+h-i} - X_{n+h-i}^{n+h-i-1}\right)|X_1,\ldots,X_n\right) \\
&= \sum_{i=1}^{n+h-1} \theta_{n+h-1,i} P\left(\left(X_{n+h-i} - X_{n+h-i}^{n+h-i-1}\right)|X_1,\ldots,X_n\right) \\
&= \sum_{i=h}^{n+h-1} \theta_{n+h-1,i} P\left(\left(X_{n+h-i} - X_{n+h-i}^{n+h-i-1}\right)|X_1,\ldots,X_n\right) \\
&= \sum_{i=h}^{n+h-1} \theta_{n+h-1,i}\left(X_{n+h-i} - X_{n+h-i}^{n+h-i-1}\right)
\end{aligned}
$$

Predicting h steps ahead using innovations (Details)

$$P(X_{n+1}|X_1,\ldots,X_n) = \sum_{i=1}^{n} \theta_{ni} \left(X_{n+1-i} - X_{n+1-i}^{n-i}\right)$$

$$P(X_{n+h}|X_1,\ldots,X_n) = \sum_{j=h}^{n+h-1} \theta_{n+h-1,j} \left(X_{n+h-j} - X_{n+h-j}^{n+h-j-1}\right)$$

$$= \sum_{i=1}^{n} \theta_{n+h-1,h-1+i} \left(X_{n+1-i} - X_{n+1-i}^{n-i}\right)$$

$(j = i + h - 1)$

Mean squared error of h-step-ahead forecasts

From orthogonality of the predictors and the error,

$\mathrm{E}\left((X_{n+h} - P(X_{n+h}|X_1,\ldots,X_n))\, P(X_{n+h}|X_1,\ldots,X_n)\right) = 0.$

That is, $\mathrm{E}\left(X_{n+h} P(X_{n+h}|X_1,\ldots,X_n)\right) = \mathrm{E}\left(P(X_{n+h}|X_1,\ldots,X_n)^2\right)$.

Hence, we can express the mean squared error as

$$
\begin{aligned}
P_{n+h}^{n} &= \mathrm{E}\left(X_{n+h} - P(X_{n+h}|X_1,\ldots,X_n)\right)^2 \\
&= \gamma(0) + \mathrm{E}\left(P(X_{n+h}|X_1,\ldots,X_n)\right)^2 \\
&\quad - 2\mathrm{E}\left(X_{n+h} P(X_{n+h}|X_1,\ldots,X_n)\right) \\
&= \gamma(0) - \mathrm{E}\left(P(X_{n+h}|X_1,\ldots,X_n)\right)^2 .
\end{aligned}
$$

Mean squared error of h-step-ahead forecasts

But the innovations are uncorrelated, so

$$
\begin{aligned}
P_{n+h}^{n} &= \gamma(0) - \mathrm{E}\left(P(X_{n+h}|X_1,\ldots,X_n)\right)^2 \\
&= \gamma(0) - \mathrm{E}\left(\sum_{j=h}^{n+h-1} \theta_{n+h-1,j}\left(X_{n+h-j} - X_{n+h-j}^{n+h-j-1}\right)\right)^2 \\
&= \gamma(0) - \sum_{j=h}^{n+h-1} \theta_{n+h-1,j}^2 \, \mathrm{E}\left(X_{n+h-j} - X_{n+h-j}^{n+h-j-1}\right)^2 \\
&= \gamma(0) - \sum_{j=h}^{n+h-1} \theta_{n+h-1,j}^2 \, P_{n+h-j}^{n+h-j-1}.
\end{aligned}
$$

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Example: Innovations algorithm for forecasting an MA(1)

Suppose that we have an MA(1) process $\{X_t\}$ satisfying

$$X_t = W_t + \theta_1 W_{t-1}.$$

Given $X_1, X_2, \ldots, X_n$, we wish to compute the best linear forecast of X_{n+1}, using the innovations representation,

$$X_1^0 = 0, \qquad X_{n+1}^n = \sum_{i=1}^{n} \theta_{ni} \left(X_{n+1-i} - X_{n+1-i}^{n-i} \right).$$

Example: Innovations algorithm for forecasting an MA(1)

An aside: The linear predictions are in the form

$$X_{n+1}^n = \sum_{i=1}^{n} \theta_{ni} Z_{n+1-i}$$

for uncorrelated, zero mean random variables Z_i. In particular,

$$X_{n+1} = Z_{n+1} + \sum_{i=1}^{n} \theta_{ni} Z_{n+1-i},$$

where $Z_{n+1} = X_{n+1} - X_{n+1}^n$ (and all the Z_i are uncorrelated).
This is suggestive of an MA representation.

Why isn't it an MA?

Example: Innovations algorithm for forecasting an MA(1)

$$\theta_{n,n-i} = \frac{1}{P_{i+1}^{i}} \left(\gamma(n-i) - \sum_{j=0}^{i-1} \theta_{i,i-j} \theta_{n,n-j} P_{j+1}^{j} \right).$$

$$P_1^0 = \gamma(0) \qquad P_{n+1}^{n} = \gamma(0) - \sum_{i=0}^{n-1} \theta_{n,n-i}^2 P_{i+1}^{i}.$$

The algorithm computes $P_1^0 = \gamma(0)$, $\theta_{1,1}$ (in terms of $\gamma(1)$); P_2^1, $\theta_{2,2}$ (in terms of $\gamma(2)$), $\theta_{2,1}$; P_3^2, $\theta_{3,3}$ (in terms of $\gamma(3)$), etc.

Example: Innovations algorithm for forecasting an MA(1)

$$\theta_{n,n-i} = \frac{1}{P_{i+1}^i}\left(\gamma(n-i) - \sum_{j=0}^{i-1} \theta_{i,i-j}\theta_{n,n-j}P_{j+1}^j\right).$$

For an MA(1), $\gamma(0) = \sigma^2(1+\theta_1^2)$, $\gamma(1) = \theta_1\sigma^2$.
Thus: $\theta_{1,1} = \gamma(1)/P_1^0$;
$\theta_{2,2} = 0$, $\theta_{2,1} = \gamma(1)/P_2^1$;
$\theta_{3,3} = \theta_{3,2} = 0$; $\theta_{3,1} = \gamma(1)/P_3^2$, etc.
Because $\gamma(n-i) \neq 0$ only for $i = n-1$, only $\theta_{n,1} \neq 0$.

Example: Innovations algorithm for forecasting an MA(1)

For the MA(1) process $\{X_t\}$ satisfying

$$X_t = W_t + \theta_1 W_{t-1},$$

the innovations representation of the best linear forecast is

$$X_1^0 = 0, \qquad X_{n+1}^n = \theta_{n1} \left(X_n - X_n^{n-1} \right).$$

More generally, for an MA(q) process, we have $\theta_{ni} = 0$ for $i > q$.

Example: Innovations algorithm for forecasting an MA(1)

For the MA(1) process $\{X_t\}$,

$$X_1^0 = 0, \qquad X_{n+1}^n = \theta_{n1}\left(X_n - X_n^{n-1}\right).$$

This is consistent with the observation that

$$X_{n+1} = Z_{n+1} + \sum_{i=1}^{n} \theta_{ni} Z_{n+1-i},$$

where the uncorrelated Z_i are defined by $Z_t = X_t - X_t^{t-1}$ for $t = 1, \ldots, n+1$.

Indeed, as n increases, $P_{n+1}^n \to \mathrm{Var}(W_t)$ (recall the recursion for P_{n+1}^n), and $\theta_{n1} = \gamma(1)/P_n^{n-1} \to \theta_1$.

Recall: Forecasting an AR(p)

For the AR(p) process $\{X_t\}$ satisfying

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + W_t,$$

$$X_1^0 = 0, \qquad X_{n+1}^n = \sum_{i=1}^{p} \phi_i X_{n+1-i}$$

for $n \geq p$. Then

$$X_{n+1} = \sum_{i=1}^{p} \phi_i X_{n+1-i} + Z_{n+1},$$

where $Z_{n+1} = X_{n+1} - X_{n+1}^n$.

The Durbin-Levinson algorithm is convenient for AR(p) processes.

The innovations algorithm is convenient for MA(q) processes.

Introduction to Time Series Analysis. Lecture 9.

1. Review: Forecasting
2. Partial autocorrelation function.
3. Recursive methods: Durbin-Levinson.
4. The innovations representation.
5. Recursive methods: Innovations algorithm.
6. Example: Innovations algorithm for forecasting an MA(1)