Introduction to Time Series Analysis. Lecture 20.

1. Review: The periodogram
2. Asymptotics of the periodogram.
3. Nonparametric spectral estimation.

Review: Periodogram

The periodogram is defined as

$$\begin{aligned} I(\nu) &= |X(\nu)|^2 \\ &= \frac{1}{n}\left|\sum_{t=1}^{n} e^{-2\pi i t \nu} x_t\right|^2 \\ &= X_c^2(\nu) + X_s^2(\nu). \end{aligned}$$

$$\begin{aligned} X_c(\nu) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \cos(2\pi t \nu) x_t, \\ X_s(\nu) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \sin(2\pi t \nu_j) x_t. \end{aligned}$$

The same as computing $f(\nu)$ from the sample autocovariance (for $\bar{x} = 0$).

Asymptotic properties of the periodogram

We want to understand the asymptotic behavior of the periodogram $I(\nu)$ at a particular frequency ν, as n increases. We'll see that its expectation converges to $f(\nu)$.

We'll start with a simple example: Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(0, \sigma^2)$ (Gaussian white noise). From the definitions,

$$X_c(\nu_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \cos(2\pi t \nu_j) x_t, \qquad X_s(\nu_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sin(2\pi t \nu_j) x_t,$$

we have that $X_c(\nu_j)$ and $X_s(\nu_j)$ are normal, with

$$\mathbf{E} X_c(\nu_j) = \mathbf{E} X_s(\nu_j) = 0.$$

Asymptotic properties of the periodogram

Also,

$$
\begin{aligned}
\mathrm{Var}(X_c(\nu_j)) &= \frac{\sigma^2}{n}\sum_{t=1}^{n}\cos^2(2\pi t\nu_j) \\
&= \frac{\sigma^2}{2n}\sum_{t=1}^{n}(\cos(4\pi t\nu_j)+1) = \frac{\sigma^2}{2}.
\end{aligned}
$$

Similarly, $\mathrm{Var}(X_s(\nu_j)) = \sigma^2/2$.

Asymptotic properties of the periodogram

Also,

$$
\begin{aligned}
\mathrm{Cov}(X_c(\nu_j), X_s(\nu_j)) &= \frac{\sigma^2}{n}\sum_{t=1}^{n}\cos(2\pi t\nu_j)\sin(2\pi t\nu_j) \\
&= \frac{\sigma^2}{2n}\sum_{t=1}^{n}\sin(4\pi t\nu_j) = 0, \\
\mathrm{Cov}(X_c(\nu_j), X_c(\nu_k)) &= 0 \\
\mathrm{Cov}(X_s(\nu_j), X_s(\nu_k)) &= 0 \\
\mathrm{Cov}(X_c(\nu_j), X_s(\nu_k)) &= 0.
\end{aligned}
$$

for any $j \neq k$.

Asymptotic properties of the periodogram

That is, if $X_1, \ldots, X_n$ are i.i.d. $N(0, \sigma^2)$ (Gaussian white noise; $f(\nu) = \sigma^2$), then the $X_c(\nu_j)$ and $X_s(\nu_j)$ are all i.i.d. $N(0, \sigma^2/2)$. Thus,

$$\frac{2}{\sigma^2} I(\nu_j) = \frac{2}{\sigma^2} \left(X_c^2(\nu_j) + X_s^2(\nu_j) \right) \sim \chi_2^2.$$

So for the case of Gaussian white noise, the periodogram has a chi-squared distribution that depends on the variance σ^2 (which, in this case, is the spectral density).

Asymptotic properties of the periodogram

Under more general conditions (e.g., normal $\{X_t\}$, or linear process $\{X_t\}$ with rapidly decaying ACF), the $X_c(\nu_j)$, $X_s(\nu_j)$ are all asymptotically independent and $N(0, f(\nu_j)/2)$.

Consider a frequency ν. For a given value of n, let $\hat{\nu}^{(n)}$ be the closest Fourier frequency (that is, $\hat{\nu}^{(n)} = j/n$ for a value of j that minimizes $|\nu - j/n|$). As n increases, $\hat{\nu}^{(n)} \to \nu$, and (under the same conditions that ensure the asymptotic normality and independence of the sine/cosine transforms), $f(\hat{\nu}^{(n)}) \to f(\nu)$. (picture)

In that case, we have

$$\frac{2}{f(\nu)} I(\hat{\nu}^{(n)}) = \frac{2}{f(\nu)} \left(X_c^2(\hat{\nu}^{(n)}) + X_s^2(\hat{\nu}^{(n)}) \right) \xrightarrow{\mathrm{d}} \chi_2^2.$$

Asymptotic properties of the periodogram

Thus,

$$
\begin{aligned}
\mathbf{E} I(\hat{\nu}^{(n)}) &= \frac{f(\nu)}{2} \mathbf{E} \left(\frac{2}{f(\nu)} \left(X_c^2(\hat{\nu}^{(n)}) + X_s^2(\hat{\nu}^{(n)}) \right) \right) \\
&\to \frac{f(\nu)}{2} \mathbf{E}(Z_1^2 + Z_2^2) = f(\nu),
\end{aligned}
$$

where Z_1, Z_2 are independent $N(0,1)$. Thus, the periodogram is asymptotically unbiased.

Asymptotic properties of the periodogram

Since we know its asymptotic distribution (chi-squared), we can compute approximate confidence intervals:

$$\Pr\left\{\frac{2}{f(\nu)} I(\hat{\nu}^{(n)}) > \chi_2^2(\alpha)\right\} \to \alpha,$$

where the cdf of a χ_2^2 at $\chi_2^2(\alpha)$ is $1-\alpha$. Thus,

$$\Pr\left\{\frac{2I(\hat{\nu}^{(n)})}{\chi_2^2(\alpha/2)} \leq f(\nu) \leq \frac{2I(\hat{\nu}^{(n)})}{\chi_2^2(1-\alpha/2)}\right\} \to 1-\alpha.$$

Asymptotic properties of the periodogram: Consistency

Unfortunately, $\mathrm{Var}(I(\hat{\nu}^{(n)})) \to f(\nu)^2 \mathrm{Var}(Z_1^2 + Z_2^2)/4$, where Z_1, Z_2 are i.i.d. $N(0,1)$, that is, the variance approaches a constant.

Thus, $I(\hat{\nu}^{(n)})$ is not a consistent estimator of $f(\nu)$. In particular, if $f(\nu) > 0$, then for $\epsilon > 0$, as n increases,

$$\Pr\left\{ \left| I(\hat{\nu}^{(n)}) - f(\nu) \right| > \epsilon \right\}$$

approaches a constant.

Asymptotic properties of the periodogram: Consistency

This means that the approximate confidence intervals we obtain are typically wide.

The source of the difficulty is that, as n increases, we have additional data (the n values of x_t), but we use it to estimate additional independent random variables, (the n independent values of $X_c(\nu_j)$, $X_s(\nu_j)$).

How can we reduce the variance? The typical approach is to average independent observations. In this case, we can take an average of “nearby” values of the periodogram, and hope that the spectral density at the frequency of interest and at those nearby frequencies will be close.

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Nonparametric spectral estimation

Define a band of frequencies

$$\left[\nu_k - \frac{L}{2n}, \nu_k + \frac{L}{2n}\right]$$

of bandwidth L/n. Suppose that $f(\nu)$ is approximately constant in this frequency band.

Consider the following *smoothed spectral estimator*. (assume L is odd)

$$\begin{aligned}\hat{f}(\nu_k) &= \frac{1}{L} \sum_{l=-(L-1)/2}^{(L-1)/2} I(\nu_k - l/n) \\ &= \frac{1}{L} \sum_{l=-(L-1)/2}^{(L-1)/2} \left(X_c^2(\nu_k - l/n) + X_s^2(\nu_k - l/n)\right).\end{aligned}$$

Nonparametric spectral estimation

For a suitable time series (e.g., Gaussian, or a linear process with sufficiently rapidly decreasing autocovariance), we know that, for large n, all of the $X_c(\nu_k - l/n)$ and $X_s(\nu_k - l/n)$ are approximately independent and normal, with mean zero and variance $f(\nu_k - l/n)/2$. From the assumption that $f(\nu)$ is approximately constant across all of these frequencies, we have that, asymptotically,

$$\hat{f}(\nu_k) \sim f(\nu_k)\frac{\chi^2_{2L}}{2L}.$$

Nonparametric spectral estimation

Thus,

$$\mathbf{E}\hat{f}(\hat{\nu}^{(n)}) \approx \frac{f(\nu)}{2L}\mathbf{E}\left(\sum_{i=1}^{2L} Z_i^2\right) = f(\nu),$$

$$\mathrm{Var}\hat{f}(\hat{\nu}^{(n)}) \approx \frac{f^2(\nu)}{4L^2}\mathrm{Var}\left(\sum_{i=1}^{2L} Z_i^2\right) = \frac{f^2(\nu)}{2L}\mathrm{Var}(Z_1^2),$$

where the Z_i are i.i.d. $N(0,1)$.

Nonparametric spectral estimation: confidence intervals

From the asymptotic distribution, we can define approximate confidence intervals as before:

$$\Pr\left\{\frac{2L\hat{f}(\hat{\nu}^{(n)})}{\chi^2_{2L}(\alpha/2)} \leq f(\nu) \leq \frac{2L\hat{f}(\hat{\nu}^{(n)})}{\chi^2_{2L}(1-\alpha/2)}\right\} \approx 1-\alpha.$$

For large L, these will be considerably tighter than for the unsmoothed periodogram. (But we need to be sure f does not vary much over the bandwidth L/n.)

Nonparametric spectral estimation

Notice the *bias-variance trade off*:
For bandwidth $B = L/n$, we have $\mathrm{Var}\hat{f}(\nu_k) \approx c/(Bn)$ for some constant c.
So we want a bigger bandwidth B to ensure low variance (*bandwidth stability*).
But the larger the bandwidth, the more questionable the assumption that $f(\nu)$ is approximately constant in the band $[\nu - B/2, \nu + B/2]$. For a larger value of B, our estimate $\hat{f}(\nu)$ will be a smoother function of ν. We have thus introduced more *bias* (lower *resolution*).

Nonparametric spectral estimation: confidence intervals

Since the asymptotic mean and variance of $\hat{f}(\hat{\nu}^{(n)})$ are proportional to $f(\nu)$ and $f^2(\nu)$, it is natural to consider the *logarithm* of the estimator. Then we can define approximate confidence intervals as before:

$$\Pr\left\{\frac{2L\hat{f}(\hat{\nu}^{(n)})}{\chi^2_{2L}(\alpha/2)} \le f(\nu) \le \frac{2L\hat{f}(\hat{\nu}^{(n)})}{\chi^2_{2L}(1-\alpha/2)}\right\} \approx 1-\alpha,$$

$$\Pr\left\{\log\left(\hat{f}(\hat{\nu}^{(n)})\right) + \log\left(\frac{2L}{\chi^2_{2L}(\alpha/2)}\right)\right.$$

$$\left.\le \log(f(\nu)) \le \log\left(\hat{f}(\hat{\nu}^{(n)})\right) + \log\left(\frac{2L}{\chi^2_{2L}(1-\alpha/2)}\right)\right\} \approx 1-\alpha.$$

The width of the confidence intervals for $f(\nu)$ varies with frequency, whereas the width of the confidence intervals for $\log(f(\nu))$ is the same for all frequencies.